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Note: Detailed derivations are required to obtain a full score for each problem.

1. (15 pts) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & a \end{pmatrix}$, $x \in \mathsf{M}_{3\times1}(\mathbb{R})$, and $b = \begin{pmatrix} 100 \\ 200 \\ c \end{pmatrix}$.

- **(a)** (5%) Find the conditions for a such that the system of equations $Ax = b$ has a unique solution.
- **(b)** (5%) Find the conditions for a and c such that $Ax = b$ has infinitely many solutions.
- **(c)** (5%) If $Ax = b$ has infinitely many solutions, is it possible to find a positive integer n such that $A^n x = b$ has a unique solution? Why or why not?

2. (10 pts) Let V be the vector space spanned by the set of functions $\{1, \cos\omega t, \sin\omega t\}$, defined on the time domain $t \in \mathbb{R}$. Assume that the angular frequency $\omega \geq 0$, and let $\beta = \{1, \cos\omega t, \sin\omega t\}$ be regarded as an ordered basis for V. Define a linear transformation $\mathsf{T} : \mathsf{V} \to \mathsf{V}$ as follows,

$$\mathsf{T}(x(t)) = m\frac{d^2x(t)}{dt^2} + r\frac{dx(t)}{dt} + kx(t),$$

where parameters m, r, k are non-negative.

- **(a)** (5%) Find the matrix representation $A = [\mathsf{T}]_\beta$.
- **(b)** (5%) If $r = 0, m \neq 0$, find the condition for ω such that $\dim(\mathsf{N}(\mathsf{T})) > 0$.

3. (9 pts) Let V be an inner product space and let T be a linear operator on V. Prove or disprove the following statement.

$$\mathsf{R}(\mathsf{T}^*)^\perp = \mathsf{N}(\mathsf{T}).$$

4. (16 pts) Given the Schur decomposition theorem as follows.

Theorem 1 *Let T be a linear operator on a finite-dimensional inner product space V. If $\det(T - tI_V)$ splits, then there exists an orthonormal basis β for V such that $[T]_\beta$ is upper triangular.*

Use this theorem to prove or disprove the following statements.

- **(a)** (8%) Let T be a normal operator on a finite-dimensional inner product space V over $\mathbb{C}$. Then there exists an orthonormal basis β for V such that $[\mathsf{T}]_\beta$ is diagonal.

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(b) (8%) Let T be a self-adjoint operator on a finite-dimensional inner product space V over $\mathbb{R}$. Then there exists an orthonormal basis β for V such that $[\mathsf{T}]_\beta$ is diagonal.

5. (5 pts) Let X and Y be two discrete random variables defined on the same sample space.

(a) (1%) Please give the condition that X and Y are statistically independent.

(b) (1%) Please give the condition that X and Y are uncorrelated.

(c) (1%) Please give the condition that X and Y are orthogonal.

(d) (2%) Please describe the memoryless property of X if X is a geometric random variable.

6. (10 pts)

(a) (5%) Show that

$$p(n) = \frac{1}{n(n+1)}, n \geq 1,$$

is a probability mass function.

(b) (5%) Let X be a random variable with probability mass function p given in part (a); find $E(X)$.

7. (10 pts) In this problem, we consider transmission of binary bits (a bit is the smallest unit of information and is either 1 or 0) over the binary symmetric channel (BSC). The BSC channel has two input symbols (0 and 1) and two output symbols (0 and 1). The channel is symmetric because the probability of receiving a 1 if a 0 is sent is the same as the probability of receiving a 0 if a 1 is sent. The conditional probability of error is denoted by p.

(a) (5%) If a bit is sent repeatedly until it is transmitted reliably (correctly), and if each time it takes 2 minutes to process it, what is the probability that the reliable transmission of a bit takes more than t minutes?

(b) (5%) The simplest error detection scheme used in data communication is **parity checking.** Usually messages sent consist of characters, each character consisting of a number of bits. In parity checking, a 1 or 0 is appended to the end of each character at the transmitter to make the total number of 1's even. The receiver checks the number of 1's in every character received, and if the result is odd it signals an error. Suppose that each bit is transmitted independently over the BSC channel. What is the probability that a 7-bit character is received in error, but the error is not detected by the parity check? It is worth noting that an extra bit is appended to the end of the 7-bit character for parity checking at the transmitter.

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8. (5 pts) Let the joint probability density function of continuous random variables X, Y be given by the following:

$$f(x,y) = \begin{cases} 2 & \text{if } 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the conditional probability density function $f_{X|Y}(x|y)$.

9. (10 pts)

(a) (5%) State and prove the weak law of large number

(b) (5%) A sequence of random variable converges to a number c in the mean-square sense if:

$$\lim_{n\to\infty} E[(X_n - c)^2] = 0$$

Show that convergence in mean-square sense (as above) implies convergence in probability (as below):

$$\lim_{n\to\infty} P(|X_n - c| \geq \epsilon) = 0.$$

10. (10 pts)
In data communication, a messeage is usually a combination of characters, and each character consists of a number of bits. Suppose that the number of bits for a character is a geometric random variable with parameter, p. Now, suppose that a message is composed of K characters, where K is a random variable with mean μ and variance σ^2. The lengths (number of bits) of characters in a message are independent of each other and of K. Lastly, it takes a message sender $\frac{1}{1000}$th of a second to send 1 bit. Find the expected value and the variance of T, i.e., the time it takes for the sender to send a message.