1. Use Laplace transforms to solve the following problem for $y(t)$, $t \geq 0$

$$\frac{d^2y}{dt^2} + y = 2t, \quad y(\frac{\pi}{4}) = \frac{\pi}{2}, \quad \frac{dy(\frac{\pi}{4})}{dt} = 0$$ (15%)

2. Consider a system governed by the following equation

$$\mathbf{Ax} = \lambda \mathbf{Bx}, \text{ where } \mathbf{A} = \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Find the eigenvalues λ_i and eigenvectors $\mathbf{x}_i$ of the above equation. (15%)

3. The equation $y'^2 - xy' + y = 0$ has the general solution $y = cx - c^2$. Find the singular solution of $y'^2 - xy' + y = 0$. [The general solution $y = cx - c^2$ represents a family of straight lines where each line corresponds a definite value of c. The envelop of $y = cx - c^2$ will be the singular solution of $y'^2 - xy' + y = 0$. Note that a family of curves $f(x, y, c) = 0$, where c is the parameter which determines different members of the family, in general, envelops a curve. The envelop of the curve family can be obtained by eliminating c from the two equations $f(x, y, c) = 0$ and $\frac{\partial f}{\partial c} = f_c(x, y, c) = 0$.] (10%)

4. Find the radius of curvature of the right-handed circular helix defined by the vector equation

$$\mathbf{r}(t) = a\cos\theta\ \mathbf{i} + a\sin\theta\ \mathbf{j} + b\theta\ \mathbf{k} \qquad a, b > 0, \qquad 0 \leq \theta < \infty.$$ (15%)

5. Verify the identity

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

by finding the Fourier series for

$$f(x)=\frac{x^2}{4} \qquad -\pi\le x\le\pi$$

at $x=\pi$. (10%)

6. (a) Solve the heat conduction problem with the method of separation of variables

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0$$

$$T(0,y)=1,\ T(\infty,y)=0,\ \partial T(x,0)/\partial y=0,\ T(x,1)=0$$ (10%)

(b) How do you solve the problem, if the boundary conditions are

$$T(0,y)=0,\ T(1,y)=1,\ \partial T(x,0)/\partial y=0,\ T(x,1)=1?$$

(Note: Answer the question (b) briefly. There is no need to solve the problem.) (5%)

7. Evaluate the following integrals where $i^2=-1$

(a) $\int_0^{\pi/6} e^{i2t}\,dt$ (6%)

(b) $\oint_C \dfrac{z\,dz}{z^2-3z+2}$ where C is the circle $|z|=3$ counterclockwise in a complex plane. (6%)

(c) $\int_0^{\infty}\dfrac{2x^2-1}{x^4+5x^2+4}dx$ [Hint: an improper integral of an even function] (8%)