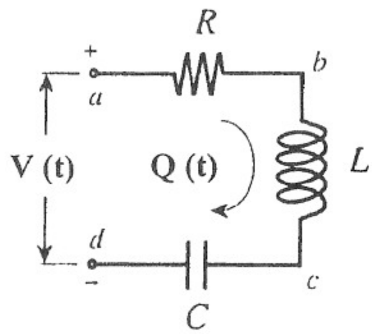
國　立　清　華　大　學　命　題　紙

96 學年度 微機電系統工程研究所 系（所）________組碩士班入學考試

科目 工程數學　科目代碼 1903　共 5 頁第 1 頁　*請在【答案卷卡】內作答

1. Referring to the electrical circuit on the left, ***R***, ***L***, and ***C*** denote the resistance, inductance, and capacitance, respectively. Suppose that a time-varying voltage $V(t)= V_0\cos\Omega t$ is applied on this circuit, leading to a charge flow $Q(t)$ accordingly.



Remark : *R: resistance, L: inductance, C: capacitance, Q(t) : charges, V(t) : voltage drop*

(a). Use Kirchhoff's voltage law to model this system *(5 pts.)*

(b). Determine the resonant frequency ω of this circuit *(5 pts.)*

(c). For the special case $\Omega=\omega$, solve (a) and plot the particular sol. $Q_p(t)$ (*15 pts.*)

(d). If $R=0$, $V(t)=V_0$, $Q(0)$ and $Q'(0)$ are known, apply Laplace transform to solve your answer in 1(a) *(10 pts.)*

Remark : *R: resistance, L: inductance, C: capacitance, Q(t) : charges, V(t) : voltage drop*

2. Jones calculus, a broadly useful mathematical technique for polarized light, includes two representations, *Jones vectors* and *Jones Matrices*, to describe the states of polarizations and the linear operation of any optical device.

(a). For example, the 2×2 Jones matrix **J** of a right circular polarizer is made of two row vectors, $e_1=[1, i]$ and $e_2=[-i, 1]$. Are e_1 and e_2 orthogonal? (5 pts.)

(b). It is possible to represent the passage of a beam of light through multiple optical devices as the multiplication. For a vertically polarized input signal $\vec{E}_i=[1,0]^T$, propagating through two devices, a linear polarizer orientated at 45^0 and then a quarter-wave plate with its fast axis vertical, what is the output signal $\vec{E}_t$? *Hint:* $\vec{E}_t = J\vec{E}_i$ *(10 pts.)*

TABLE 1. Jones Matrices of Common Optical Devices

Vertical Linear Polarizer	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$	Right Circular Polarizer	$\frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$
Horizontal Linear Polarizer	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	Left Circular Polarizer	$\frac{1}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$
Linear Polarizer at 45°	$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	Quarter-wave plate, fast axis vertical	$e^{i\pi/4}\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$
Lossless fiber transmission	$\begin{bmatrix} e^{i\phi}\cos\theta & -e^{-i\psi}\sin\theta \\ e^{i\psi}\sin\theta & e^{-i\phi}\cos\theta \end{bmatrix}$	Quarter-wave plate, fast axis horizontal	$e^{i\pi/4}\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$

3. **Partial differential equation and eigen value problem**. In the figure 3, there are two discs with the same rotational inertia J. The two discs are connected by a shaft. The length of the shaft is ℓ with a constant density of ρ. Its torsional stiffness is GI_p. The shaft is supported by two ideal bearings and therefore can rotate freely. The torsional vibration of the shaft can be described through the angular function $\phi(x, t)$. The behavior equation in the dimensionless form is :

$$\omega_0^{\ 2}\,\varphi_{\xi\xi} - \varphi_{tt} = 0$$

The boundary conditions are:

$$\omega_0^{\ 2}\,\varphi_{\xi}(0,t) - \varepsilon\varphi_{tt}(0,t) = 0$$

$$\omega_0^{\ 2}\,\varphi_{\xi}(1,t) + \varepsilon\varphi_{tt}(1,t) = 0$$

Where

$$\xi = \frac{x}{\ell}, \quad \frac{d}{d\xi} = \ell\frac{d}{dx}$$

$$\omega_0^{\ 2} = \frac{G}{\rho\cdot\ell}, \qquad \varepsilon = \frac{J}{\rho\cdot I_p\cdot\ell}$$

$$\varphi_{\xi} \equiv \frac{\partial\varphi}{\partial\xi}, \quad \varphi_{\xi\xi} \equiv \frac{\partial^2\varphi}{\partial\xi^2}, \quad \varphi_t \equiv \frac{\partial\varphi}{\partial t}, \quad \varphi_{tt} \equiv \frac{\partial^2\varphi}{\partial t^2}$$

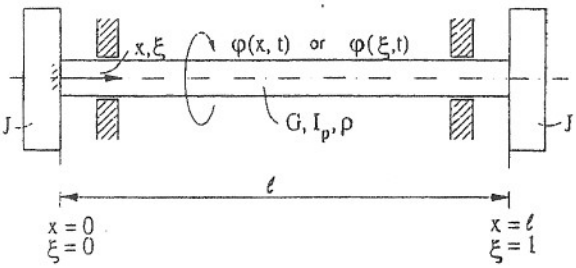


Fig. 3 Partial differential equation and eigenvalue problem.

(a) Assume that $\varphi(\xi,\mathrm{t}) = \Phi(\xi)\cdot \mathrm{T(t)}$ and $\frac{\ddot{\mathrm{T}}}{\mathrm{T}} = -\lambda^2\omega_0{}^2$, please find the time free behavior equation and the boundary conditions by the method of separating variables or the product method. (4pts)

(b) Please write the eigenvalue equation of the system from the time free behavior equation and boundary conditions. (4pts)

(c) Please find the complete eigenvalues, when the both J=0. (4pts)

(d) From the first nonzero eigenfunction $\Phi_1(\xi)$, please find the behavior equation with time $\mathrm{T_1(t)}$ and the corresponding vibration equation of the system $\varphi_1(\xi,\mathrm{t})$ (4pts)

4. **Fourier Transform and System.** A system consists of a mass 'm' and a damping '2D' as shown in the figure 4. This dynamic system is fixed at the left end. When t = 0, the system is in the rest. An external force x(t) is applied to the system and $\mathrm{x(t)} = \exp(-2\mathrm{Dt})\,\sigma(\mathrm{t})$, where $\sigma(\mathrm{t}) = 0$, when $\mathrm{t} < 0$, and $\sigma(\mathrm{t}) = 1$, when $\mathrm{t} \geq 0$. as shown in the figure.

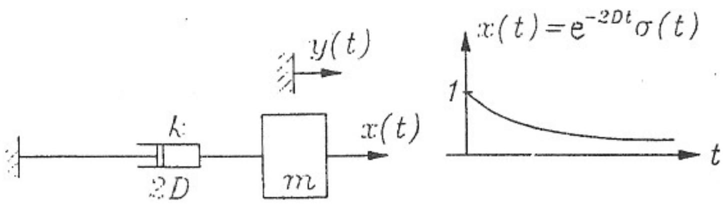


Fig. 4 Question for Fourier Transform and System

The behavior of the system can be described as follows:

$$\ddot{y} + 2D\dot{y} = x(t)$$

$$Y(i\omega) = F(i\omega) \cdot X(i\varpi)$$

Please discuss the dynamic behavior of this system in the frequency domain using Fourier transformation:

(a) Please write the complex input spectrum $X(i\omega)$. (4 pts)

(b) Please find the relationship in the complex form of the system $F(i\omega)$ and $X(i\omega)$ (4 pts)

(c) Please find the complex output spectrum $Y(i\omega)$ (4pts)

(d) Please calculate in the spectrum density $S(\omega)=|Y(i\omega)|$ and the value $S(0)$ and $S(\infty)$. (4pts)

(e) Please calculate the solution in the time domain $y(t)$ for $t>0$ through the inverse Fourier transform. (4pts)

Remark:

$$\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{i\omega(a+i\omega)^2} d\omega = \begin{pmatrix} 0 & , t<0 \\ \dfrac{2\pi \cdot \gamma(2, at)}{a^2 \cdot \Gamma(2)} & , t>0 \end{pmatrix}$$

$$\text{with } \Gamma(2) = \int_0^{\infty} te^{-t} dt, \quad \gamma(2, at) = \int_0^{at} \bar{t}e^{-\bar{t}} d\bar{t},$$

$$\int xe^{-x} dx = \frac{-(x+1)}{e^x}$$

5. **Hamilton's Principle and Variation Principle (Mathematics for Conservative Systems).** In the figure 5, you can see a rod fixed at the left end. This rod has a constant cross section "A" and a homogeneous mass distribution "μ" (density). Through the elasticity "E" (Young's Modulus), the rod can be extended or compressed back and forth (displacement vibration) at different position of the rod "u(x, t)". At the right end, the rod is coupled with a mass "m" through a spring "c" and a damping "k" as shown in the figure. "P(t)" is the external force applied on the mass "m". The motion of the mass "m" is described through the absolute coordinate y. When y = 0, both the rod and the spring are in the relax condition and the mass is in the rest.

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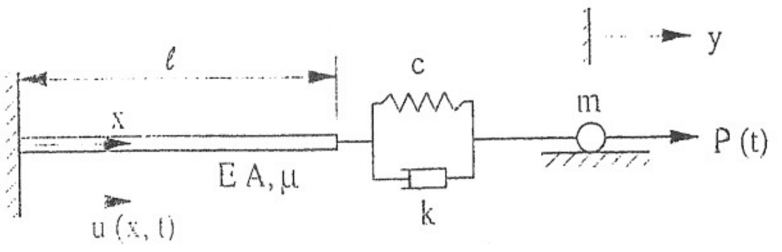

Fig. 5 Hamilton's and Variation Principle

Hamilton Principle is described as follows:

$$\delta\int_{t_0}^{t_1}(T - Vi - Ve)dt + \int_{t_0}^{t_1}\delta W dt = 0$$

Where T is the kinetic energy of the system, Vi is the internal potential energy, Ve is the potential energy results from the external force and δW is the virtual work. In order to describe the vibration of this system, for example, at different position of the rod u(x, t) or the motion of the mass 'm', please follow the questions and answer them using Hamilton's and the variational principles.

(a) Please find the T, Vi, Ve and δW of this system. (2pts)

(b) Please induce the Hamilton variation principle to obtain the extended variations and production integrations. (3pts)

(c) From (b), please find the differential equation which describes the vibration of the rod u(x, t) (3pts)

(d) From (b), please find the differential equation which describes the motion of the mass 'm'. (3pts)

(e) Please use the boundary conditions to find the force at the end of the rod $F = EAu_x(\ell, t)$. (3pts)

Remark:

$u(0,t)=u_x(0,t)=0$, $u_x = \dfrac{\partial u}{\partial x}$, $u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$

Integration by parts: $\int_a^b f(x)g'(x)dx = f(x)g(x)\big|_a^b - \int_a^b f'(x)g(x)dx$