1. Let $f = 4x^2 + xy^2 + 9y^3z^2$ (scalar function) and $V = xz\mathbf{i} + (x-y)^2\mathbf{j} + 2x^2yz\mathbf{k}$ (vector function).
   Find
   (a) $\nabla^2 f$ (5%)
   (b) $curl(grad\ f)$ (5%)
   (c) $\nabla f \bullet curl V$ (5%)

2. Solve the following first-order differential equation for $u(t)$:

   $$\frac{du}{dt} = \exp(t+u), \quad u(0) = 1$$ (10%)

3. For a matrix $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$,

   (a) Find the 2x2 matrix P and D, such as $P^{-1}AP=D$ where D is a diagonal matrix. (5%)
   (b) Find the eigenvalues and the corresponding eigenvectors for $f(A)$, where $f(x) = 5x + 2$. (5%)

4. Use Laplace transforms to solver the following equations for $y(t)$

   (a) $\frac{d^2y}{dt^2} + y = \cos(2t)$, where $t \geq 0$, $y(0) = 1$, and $\frac{dy(0)}{dt} = 0$. (10%)

   (b) $y(t) = 6t + \int_0^t y(t-s)\sin(s)ds$, $t \geq 0$. (10%)

5. Evaluate the following integrals by using Fourier Transform.

   (a) $\int_0^\infty \frac{dx}{x^2+1}$ (5%)

   (b) $\int_0^\infty \frac{\sin(ax)}{\sinh(bx)}dx$ (10%)

   (Hint: $e^{ikx} = \cos kx + i\sin kx$ and $\sinh(bx) = \frac{\exp(bx) - \exp(-bx)}{2}$)

6. Solve the following second-order differential equation for $u(t)$:

   $$t^2\frac{d^2u}{dt^2} + t\frac{du}{dt} + 4u = \sin[\ln(t)]$$ (10%)

7. Below is so called the one-dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

(a) Find the deflection $u(x,t)$ of the vibrating string based on the following conditions, (14%)

- Boundary conditions: $u(0,t) = 0$ and $u(L,t) = 0$ for all t.

- Initial conditions: $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}\big|_{t=0} = g(x)$.

(b) Find the deflection $u(x,t)$ by using the derived result from (a) and the following functions and parameters, (6%)

- $c = 1$ and $L = \pi$.
- $f(x) = 0$ and $g(x) = 0.1\sin(2x)$.