國　立　清　華　大　學　命　題　紙

八十七學年度 工業工程 系（所） 工工甲 組碩士班研究生入學考試

科目 線性代數 科號 3302 共 3 頁第 1 頁 *請在試卷【答案卷】內作答

1. Yes or No (每題 3 分，答錯一題倒扣 2 分，最高得 30 分，最低得 0 分，無負分)　　(30%)

(1-1) **A** is an $m \times n$ matrix. The following statements are always logically equivalent (i.e., both true or both false).

Statement 1.　　The columns of **A** span $\mathbf{R}^m$.

Statement 2:　　**A** has a pivot position in every row.

(1-2) $\mathbf{u} = \begin{bmatrix} 8 \\ 2 \\ 4.5 \end{bmatrix}$ and $\mathbf{A} = \begin{bmatrix} 4 & 3 & 5 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix}$. **u** is in the subset of $\mathbf{R}^3$ spanned by the columns of **A**.

(1-3) A homogeneous system has a nontrivial solution if and only if the system has at least one free variable.

(1-4) The general solution of **Ax**=**b** has the form

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 1.5 \\ 0 \\ 1 \end{bmatrix}$$

The solution set is a line in $\mathbf{R}^3$ through $\begin{bmatrix} 1.5 \\ 0 \\ 1 \end{bmatrix}$ parallel to $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(1-5) The solution set in $\mathbf{R}^3$ of $x_1 + 3x_2 - 8x_3 = 0$ is a line passing through the origin.

(1-6) If $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$ are linearly independent vectors in $\mathbf{R}^4$, then the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is also linearly independent.

(1-7) A transformation, $\mathbf{x} \mapsto \mathbf{Ax}$, reflects points through the y-axis. In $\mathbf{R}^2$, **A** must be

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(1-8) The vector space $\mathbf{R}^2$ is not a sub-space of $\mathbf{R}^3$.

(1-9) **V** is the second quadrant in the *xy*-plane; that is, $\mathbf{V} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \leq 0 \right\}$.

Thus, **V** is a vector space.

(1-10) **A** is a 4×7 matrix and has four pivot columns. Thus, Col **A** = $\mathbf{R}^4$ and Nul **A** = $\mathbf{R}^3$.

2. Use the Gauss-Jordan method to determine whether each of the following linear systems has no solution, a unique solution, or an infinite number of solutions. Indicate the solution (if any exist). (15%)

(a)
$$\begin{aligned} x_1 + x_2 \qquad + x_4 &= 3 \\ x_2 + x_3 \qquad &= 4 \\ x_1 + 2x_2 + x_3 + x_4 &= 8 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + x_2 + x_3 &= 4 \\ x_1 + 2x_2 \qquad &= 6 \end{aligned}$$

(c)
$$\begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 + x_2 &= 3 \\ 3x_1 + 2x_2 &= 4 \end{aligned}$$

3. Each year, 20% of all untenured University faculty become tenured (永久職), 5% quit, and 75% remain untenured. Each year, 90% of all tenured faculty remain tenured and 10% quit or retire. Let U_t be the number of untenured faculty at the beginning of year t, and T_t be the number of tenured faculty at the beginning of year t. Use matrix multiplication to relate the vector $[U_{t+1}, T_{t+1}]$ to the vector $[U_t, T_t]$. (10%)

4. Let

$$\mathbf{A} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$$

(4-1) For what value of a, b, c, and d will $\mathbf{A}^{-1}$ exist? (5%)

(4-2) If $\mathbf{A}^{-1}$ exists, find it. (5%)

5. **(5-1)** **A** is an $m \times n$ matrix and **b** is in $\mathbf{R}^m$. Give the following definition:

"a **least-square solution** of $\mathbf{Ax} = \mathbf{b}$" (5%)

(5-2) Find all least-square solutions of the following system. (10%)

$$x + y = 2$$
$$x + y = 4$$

6. **(6-1)** **A** and **B** are $n \times n$ matrices. Give the following definitions:

(i) **characteristic polynomial** of **A**

(ii) **A** is **similar** to **B** (10%)

(6-2) Prove that if $n \times n$ matrices **A** and **B** are similar, then they have the same characteristic polynomial and hence the same eigenvalues. (10%)