

科目 應用數學 類組別 037 共 2 頁第 1 頁 *請在試卷答案卷(卡)內作答

1. A damping motion of a unit mass on a spring, $y(t)$, is governed by a second-order differential equation $\ddot{y} + c\dot{y} + ky = 0$, where c is the damping factor and k is the spring constant, and both are constants.

- Please convert this 2nd-order differential equation to a 1st-order differential system. Write down the system in a matrix form. (10%)
- Assuming that $c = 6$ and $k = 8$, please show two eigenvalues are -2 and -4 for the solution of this vibrating system. (5%)
- Assuming that $c = 0$ and $k = 4$, please show two eigenvalues are $2i$ and $-2i$ for the system ($i = \sqrt{-1}$). (5%)
- Assuming that $c = 2$ and $k = 2$, please show two eigenvalues are $-1 \pm i$ for the motion. (5%)
- Considering the real and imaginary parts of these eigenvalues, please briefly explain the types of critical points in the phase plane, i.e. the $y\dot{y}$ -plane, for the above three cases are (b) improper node (Fig. 1), (c) center (Fig. 2) and (d) spiral point (Fig. 3), respectively. (15%)

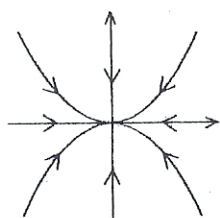


Figure 1

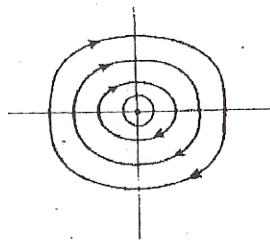


Figure 2

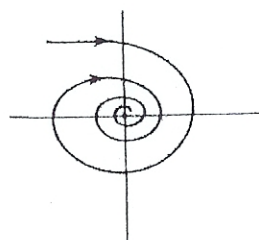


Figure 3

2. For an infinite bar, one-dimensional heat equation $\frac{\partial u(x,t)}{\partial t} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$ with the initial condition $u(x,0) = f(x)$ has an solution of the error function form

科目 應用數學 類組別 037 共 2 頁第 2 頁 *請在試卷答案卷(卡)內作答

$$u(x, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x + 2cz\sqrt{t}) e^{-z^2} dz.$$

- (a) Please derive the solution shown above by Fourier integrals. (20%)
- (b) If $f(x) = 1$ when $x > 0$ and $f(x) = 0$ when $x < 0$, please show that the solution

$$u(x, t) = \frac{1}{\sqrt{\pi}} \int_{-\frac{x}{2c\sqrt{t}}}^{\infty} e^{-z^2} dz \quad \text{for } t > 0. \quad (10\%)$$

3. For a non-homogeneous differential equation $y'' + p(x)y' + q(x)y = r(x)$, please derive the following solution by the method of variation of parameters

$$y_p(x) = -y_1 \int \frac{y_2 r}{y_1 y_2' - y_2 y_1'} dx + y_2 \int \frac{y_1 r}{y_1 y_2' - y_2 y_1'} dx$$

where y_1, y_2 form a basis of solutions of the homogeneous equation $y'' + p(x)y' + q(x)y = 0$. (20%)

4. Assuming sufficient differentiability, please show that:

(a) $\nabla \times (\nabla f) = \vec{0}$, here f is a scalar function. (5%)

(b) $\nabla \cdot (\vec{u} \times \vec{v}) = \vec{v} \cdot \nabla \times \vec{u} - \vec{u} \cdot \nabla \times \vec{v}$. (5%)