

科目 高等微積分 類組別 032 共 1 頁第 1 頁 *請在試卷答案卷(卡)內作答

- (1) (20%) In the following each statement, is it true or false? If it is true, prove it. If your answer is false, give an example.
- (a) If x_n satisfies $|x_{n+1} - x_n| < \frac{1}{5^n}$, then x_n converges.
 - (b) If x_n satisfies $|x_{n+1} - x_n| < \frac{1}{\sqrt{n}}$, then x_n is convergent.
 - (c) If x_n is a monotone increasing sequence such that $x_{n+1} - x_n \leq \frac{1}{n}$, then x_n converges.

- (2) (20%) Let $Q^c = R \setminus Q$, where Q is the set of all rational numbers.

- (a) Is $Q^c \cap [0, 1]$ a compact set? Prove your answer.
- (b) Let

$$f(x) = \begin{cases} x, & \text{if } x \in Q^c; \\ p \sin \frac{1}{q}, & \text{if } x = \frac{p}{q}, \end{cases}$$

where p and q are integers such that $(p, q) = 1$. At what points is f continuous?

- (c) Let

$$g(x) = \begin{cases} 0 & \text{if } x \in Q^c \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms.} \end{cases}$$

Prove that f is continuous on Q^c and discontinuous otherwise.

- (3) (20%) Compute the following limits.

- (a) $\lim_{n \rightarrow \infty} \sum_{k=1}^n (n^2 + k^2)^{-\frac{1}{2}}$;
- (b) $\lim_{n \rightarrow \infty} \frac{1^p + 2^p + \dots + n^p}{n^{p+1}}$, where $p > 0$.

- (4) (20%) Let $F(x, y)$ be a function of class C^1 on some neighborhood of a point $(a, b) \in R^n \times R$. Suppose that $F(a, b) = 0$ and the partial derivative $\partial_y F(a, b) \neq 0$. Prove that there exist positive numbers r_0, r_1 such that the following conclusions (A) and (B) are valid:

- (A) For each x in the ball $\|x - a\| < r_0$ there is a unique y such that $|y - b| < r_1$ and $F(x, y) = 0$. We denote this y by $f(x)$; in particular, $f(a) = b$.
- (B) The function f thus defined for $\|x - a\| < r_0$ is of class C^1 , and its partial derivatives are given by

$$\partial_j f(x) = -\frac{\partial_j F(x, f(x))}{\partial_y F(x, f(x))}, j = 1, 2, \dots, n.$$

- (C) Suppose $F(x, y)$ is a C^1 function such that $F(0, 0) = 0$. What conditions on F will guarantee that the equation $F(F(x, y), y) = 0$ can be solved for y as a C^1 function of x near $(0, 0)$?

- (5) (20%) Let $f_n(x) = xe^{-nx}$, $x \in [0, \infty)$, $n = 0, 1, 2, \dots$

- (a) Show that $f(x) = \sum_{n=0}^{\infty} f_n(x)$ exists. Compute f explicitly.
- (b) Is f continuous?
- (c) Find a suitable set on which the convergence is uniform.
- (d) May we differentiate term by term on $(0, \infty)$? Why?