Linear Algebra

(20%) each

(a) Let A be a real $m \times n$ matrix, row space U, column space V be the subspaces of R^n, R^m spanned by the row vectors and column vectors respectively. Let f be a linear mapping defined by : $x \to Ax$. Prove that f defined a linear isomorphism between U and V.

(b) Let A be a real m by n matrix. Prove or disprove that there are two sets of orthonormal basis $\{u_i : i = 1, ..., r\}$, $\{v_j : j = 1, .., .r\}$ of the row space and the column space respectively such that $A \cdot u_i = c_i \cdot v_i$ and $c_i > 0$ for all $i = 1, ..., r$.

(c) Let A^T be the transpose of the matrix A. Prove or disprove the following statements:

(I) rank($A^T \cdot A$) =rank(A) for all real matrix A,
(II) rank($A^T \cdot A$) =rank(A) for all complex matrix A,
(III) rank($A^T \cdot A$) =rank(A) for all matrix over binary field $\{0, 1\}$.

(d) Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 6 \\ 1 & 4 & 5 & 6 & 7 \end{bmatrix}$ Prove or disprove that we can find a 5×4 matrix B such that the rank of BA is :

(I) 1
(II) 3
(III) 5

(e) Let A be a real square matrix satisfies $A^2 - 3A - 2I = 0$. Prove or disprove that A is orthogonally diagonalizable.