一. 填充題(每題八分)

1. Let $H(x) = \int_0^{x^2} \frac{dt}{1+t^3}$ and $L(x) = \int_0^{x} \frac{dt}{1+t^3}$, then $H'(2) - L'(4) =$ 甲.

2. Let $f(x) = x^{\frac{1}{x}}$ on $(0, e)$, then $f(x)$ is one-to-one. Let $g(x)$ be the inverse funciton of $f(x)$. Find $g'(\sqrt{2}) =$ 乙.

3. Let $y = \tan^{-1}\sqrt{x^3+1}$, then $\frac{dy}{dx} =$ 丙.

4. $\int e^t \sin t dt =$ 丁.

5. The interval of convergence (including endpoint(s) when valid) of $\sum_{n=1}^{\infty} \frac{n}{1+n^2} x^n =$ 戊.

二. 計算與證明題(每題十二分)

1. Compute $\lim_{x\to\infty} \frac{1}{x \ln x} \int_1^x \ln t dt$.

2. Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms. Show that if $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges.

3. Compute $\iint_R \frac{\sin x}{x} dxdy$ where R is the region bounded by the curves $y = 0, y = x$ and $x = 1$.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function. Let $\mu = (\frac{3}{5}, \frac{4}{5})$ and $\nu = (\frac{-4}{5}, \frac{3}{5})$ be two unit vectors at the point $p = (1, 1)$. Suppose that $f'_\mu(p) = 3$ and $f'_\nu(p) = 2$. Find $\nabla f(p)$. (**Note** that $f'_\mu(p)$ and $f'_\nu(p)$ are the directional derivative of f at p in the direction μ and ν, respectively.)

5. A snake is moving along the path $y = \frac{1}{x}$ in the x-y plane. Suppose that at time $t > 0$, its head is at the position $(4t, \frac{1}{4t})$ and its tail is at $(t, \frac{1}{t})$. For $t > 0$, find the time t such that the snake has shortest arc length.