

一. 填充題(每題八分)

1. Let $H(x) = \int_0^{x^2} \frac{dt}{1+t^3}$ and $L(x) = \int_0^x \frac{dt}{1+t^3}$, then $H'(b) - L'(b^2) =$ 甲.

2. Let $f(x)$ be a continuous and decreasing function, and let g be the inverse function of f . If $f(2) = 1$ and $f(4) = 0$ and $\int_2^4 f(x)dx = 1$, then $\int_0^1 g(x)dx =$ 乙.

3. Let $y = \tan^{-1} \sqrt{x^3 + 1}$, then $\frac{dy}{dx} =$ 丙.

4. $\int x^3 \sqrt{1+x^2} dx =$ 丁.

5. The interval of convergence (including end point(s) when valid) of $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n (x+1)^n =$ 戊.

二. 計算與證明題(每題十二分)

1. Compute $\lim_{x \rightarrow \infty} \frac{1}{x \ln x} \int_1^x \ln t dt$.

2. Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms. Show that if $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges.

3. Compute $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} z \sqrt{4-x^2-y^2} dz dy dx$.

4. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a differentiable function. Let $\mu = \left(\frac{5}{13}, \frac{12}{13}\right)$ and $\nu = \left(\frac{3}{5}, -\frac{4}{5}\right)$ be two unit vectors at the point $p = (1, 2)$. Suppose that $f'_\mu(p) = 1$ and $f'_\nu(p) = 2$. Find $\nabla f(p)$. (Note that $f'_\mu(p)$ and $f'_\nu(p)$ are the directional derivative of f at p in the direction μ and ν , respectively).
5. A snake is moving along the path $y = \frac{1}{x}$ in the x - y plane. Suppose that at time $t > 0$, its head is at the position $(2t, \frac{1}{2t})$ and its tail is at $(t, \frac{1}{t})$. For $t > 0$, find the time t such that the snake has shortest arc length.