一. 填充題(每題八分)

1. Let $H(x) = \int_0^{x^2} \frac{dt}{1+t^3}$ and $L(x) = \int_0^{x} \frac{dt}{1+t^3}$, then $H'(b) - L'(b^2) =$ __甲__.

2. Let $f(x)$ be a continuous and decreasing funciton, and let $g(x)$ be the inverse function of f. If $f(2) = 1$ and $f(4) = 0$ and $\int_2^4 f(x)dx = 1$, then $\int_0^1 g(x)dx =$ __乙__.

3. Let $y = \tan^{-1}\sqrt{x^3+1}$, then $\frac{dy}{dx} =$ __丙__.

4. If $f(x)$ is a continuous and decreasing function such that $\int_2^4 f(x)dx = 1$, then which of the following is always true? __丁__
(a) $f(4) > 0.1$ (b) $f(4) < 0.2$ (c) $f(2) < 0.3$
(d) $f(2) > 0.4$

5. The interval of convergence (including end point(s) when valid) of $\sum_{n=1}^{\infty}(1+\frac{1}{n})^n(x+1)^n =$ __戊__.

二. 計算與證明題(每題十二分)

1. Compute $\lim_{x\to\infty} \frac{1}{x\ln x}\int_1^x \ln t\,dt$.

2. Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms. Show that if $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges.

3. Compute the volume of the solid T bounded above by $z = \sqrt{4 - x^2 - y^2}$ and below by $z = x^2 + y^2$.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function. Let $\mu = (\frac{12}{13}, \frac{-5}{13})$ and $\nu = (\frac{3}{5}, \frac{4}{5})$ be two unit vectors at the point $p = (0, 1)$. Suppose that $f'_\mu(p) = 2$ and $f'_\nu(p) = -1$. Find $\nabla f(p)$. (**Note** that $f'_\mu(p)$ and $f'_\nu(p)$ are the directional derivative of f at p in the direction μ and ν, respectively).

5. A snake is moving along the path $y = \dfrac{1}{\sqrt{x}}$ in the x-y plane. Suppose that at time $t > 0$, its head is at the position $(2t, \dfrac{1}{\sqrt{2t}})$ and its tail is at $(t, \dfrac{1}{\sqrt{t}})$. For $t > 0$, find the time t such that the snake has shortest arc length.