注意：考試開始鈴響前，不得翻閱試題，並不得書寫、畫記、作答。

國立清華大學 114 學年度碩士班考試入學試題

系所班組別：數學系

科 目 代 碼 ：0102

考 試 科 目 ：線性代數

—作答注意事項—

1. 請核對答案卷（卡）上之准考證號、科目名稱是否正確。
2. 考試開始後，請於作答前先翻閱整份試題，是否有污損或試題印刷不清，得舉手請監試人員處理，但不得要求解釋題意。
3. 考生限在答案卷上標記「由此開始作答」區內作答，且不可書寫姓名、准考證號或與作答無關之其他文字或符號。
4. 答案卷用盡不得要求加頁。
5. 答案卷可用任何書寫工具作答，惟為方便閱卷辨識，請儘量使用藍色或黑色書寫；答案卡限用 2B 鉛筆畫記；如畫記不清（含未依範例畫記）致光學閱讀機無法辨識答案者，其後果一律由考生自行負責。
6. 其他應考規則、違規處理及扣分方式，請自行詳閱准考證明上「國立清華大學試場規則及違規處理辦法」，無法因本試題封面作答注意事項中未列明而稱未知悉。

國立清華大學 114 學年度碩士班考試入學試題

系所班組別：數學系碩士班數學組、應用數學組

考試科目（代碼）：線性代數（0102）

 ★請在 [答案卷] 作答

It is required to show your work in all problems.

1. (15 %) Let V be a finite-dimensional vector space over a field F, and let W be a subspace of V. Show that there exists a linear transformation $T\colon V \to V$ whose null space and range are both W if and only if $\dim V = 2\dim W$.

2. (20 %) Let A be an $m \times n$ matrix and B be an $n \times m$ matrix over a field F, and let $C = AB$. Show that if $\det C = 0$ and $\operatorname{rank} B = m$, then the system of homogeneous linear equations $Cx = 0$ has a solution $x = v \in F^m$ such that $Bv \neq 0$.

3. (15 %) Let A be a real $m \times n$ matrix, A^t be the transpose of A, and $b \in \mathbb{R}^m$, $v \in \mathbb{R}^n$ be column vectors. Show that if the system of linear equations $Ax = b$ has no solution, and $x = v$ is a solution to the system of linear equations $(A^tA)x = A^tb$, then $\|b\| > \|Av\|$.

4. (15 %) Let V be an n-dimensional vector space over a field F. Show that if $T\colon V \to V$ is a linear operator such that the largest number of linearly independent eigenvectors of T in V is $n-1$, then the characteristic polynomial of T splits over F and has a repeated root.

5. (15 %) Find the number of 3×3 real orthogonal matrices A such that $A^4 = I$, $A^2 \neq I$, and $\begin{pmatrix}1\\2\\3\end{pmatrix}$ is an eigenvector of A. (You do **NOT** need to explicitly compute A; you just need to explain how many such A exist.)

6. (20 %) Find all 3×3 real symmetric matrices A such that the characteristic polynomial of A is $(t-1)(t-2)^2$ and the sum of the column vectors of A is $\begin{pmatrix}1\\1\\1\end{pmatrix}$. You do need to explicitly compute every entry of all such matrices A to earn full credits.