注意：考試開始鈴響前，不得翻閱試題，
並不得書寫、畫記、作答。

國立清華大學 114 學年度碩士班考試入學試題

系所班組別：數學系

科目代碼：0101

考試科目：高等微積分

—作答注意事項—

1. 請核對答案卷（卡）上之准考證號、科目名稱是否正確。
2. 考試開始後，請於作答前先翻閱整份試題，是否有污損或試題印刷不清，得舉手請監試人員處理，但不得要求解釋題意。
3. 考生限在答案卷上標記「由此開始作答」區內作答，且不可書寫姓名、准考證號或與作答無關之其他文字或符號。
4. 答案卷用盡不得要求加頁。
5. 答案卷可用任何書寫工具作答，惟為方便閱卷辨識，請儘量使用藍色或黑色書寫；答案卡限用 2B 鉛筆畫記；如畫記不清（含未依範例畫記）致光學閱讀機無法辨識答案者，其後果一律由考生自行負責。
6. 其他應考規則、違規處理及扣分方式，請自行詳閱准考證明上「國立清華大學試場規則及違規處理辦法」，無法因本試題封面作答注意事項中未列明而稱未知悉。

國立清華大學 114 學年度碩士班考試入學試題

系所班組別: 數學系碩士班數學組、應用數學組

考試科目（代碼）: 高等微積分（0101）

　★請在 [答案卷] 作答

It is required to show your work in all problems.

1. Give the reasons of your answers to the following:

   (a) (5%) Is $[0,1)$ an open subset of $\mathbb{R}$?

   (b) (5%) Is $[0,1)$ an open subset of $[0,1]$?

   (c) (5%) Is $[0,1)$ a closed subset of $[-1,1)$?

   (d) (5%) Is $[0,1)$ a compact subset of $[-1,1)$?

2. Suppose that $f(x,y,z)=\frac{1}{\sqrt{x^2+y^2+z^2}}$.

   (a) (10%) Is the function f uniformly continuous on $\mathbb{R}^3\setminus\{(0,0,0)\}$.

   (b) (5%) Show that there exists a number $M>0$ such that $\|f(a)-f(b)\|\le M\|a-b\|$ for all $a,b\in V=\{(x,y,z)\ :\ x^2+y^2+z^2\ge 1\}$.

   (c) (5%) Is the function f uniformly continuous on V?

3. (10%) Define a real function f on $(-1,\infty)$ by

$$f(x)=\begin{cases} 2x^2-1, & -1<x<0 \\ \frac{x^2-1}{x^2+1}, & 0\le x<\infty \end{cases}$$

   Let m be the infimum of the image of f. Show that $m=-1$.

4. (10%) If $f(x)=0$ for all irrational x, $f(x)=1$ for all rational x, prove that f is not Riemann-Integrable on $[a,b]$ for any $a<b$.

5. (10%) Use differentials to approximate the value $(5.97)\sqrt[4]{16.03}$.

6. Let $B(a)$ be an open ball of $\mathbb{R}^n$ with center a, and let $f: B(a)\to\mathbb{R}$ have continuous second partial derivatives on $B(a)$. Given $u\in\mathbb{R}^n$, consider the function g defined by $g(t)=f(a+tu)$ on an open interval I of 0 such that $a+tu\in B(a)$ for all $t\in I$. Show that

   (a) (5%) $g'(0)=\sum_{i=1}^n \frac{\partial f}{\partial x_i}(a)u_i$, and

   (b) (5%) $g''(0)=\sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i\partial x_j}(a)u_iu_j$,
   where $u=(u_1,u_2,\cdots,u_n)\in R^n$ and $x=(x_1,x_2,\cdots,x_n)$ is the coordinates of $\mathbb{R}^n$.

   (c) (10%) If a is a point of relative minimum of f, show that $\sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i\partial x_j}(a)u_iu_j\ge 0$ for all $u=(u_1,u_2,\cdots,u_n)\in R^n$.

國立清華大學 114 學年度碩士班考試入學試題

系所班組別: 數學系碩士班數學組、應用數學組

考試科目（代碼）: 高等微積分（0101）

 ⋆ 請在 [答案卷] 作答

7. (10%) Find conditions on a point (u_0, v_0, x_0, y_0) such that there exists real valued functions $x(u,v)$ and $y(u,v)$ which are continuously differentiable near (u_0, v_0) and satisfy the following system of equations

$$ux^2 + vy^2 - 3uv = 5$$
$$uy^2 + vx^2 + 3uv = 11.$$

Prove that the solutions satisfy $x^2 + y^2 = 16/(u+v)$.