注意：考試開始鈴響前，不得翻閱試題，並不得書寫、畫記、作答。

國立清華大學 113 學年度碩士班考試入學試題

系所班組別：數學系

科 目 代 碼 ：0102

考 試 科 目 ：線性代數

－作答注意事項－

1. 請核對答案卷（卡）上之准考證號、科目名稱是否正確。
2. 考試開始後，請於作答前先翻閱整份試題，是否有污損或試題印刷不清，得舉手請監試人員處理，但不得要求解釋題意。
3. 考生限在答案卷上標記「由此開始作答」區內作答，且不可書寫姓名、准考證號或與作答無關之其他文字或符號。
4. 答案卷用盡不得要求加頁。
5. 答案卷可用任何書寫工具作答，惟為方便閱卷辨識，請儘量使用藍色或黑色書寫；答案卡限用 2B 鉛筆畫記；如畫記不清（含未依範例畫記）致光學閱讀機無法辨識答案者，其後果一律由考生自行負責。
6. 其他應考規則、違規處理及扣分方式，請自行詳閱准考證明上「國立清華大學試場規則及違規處理辦法」，無法因本試題封面作答注意事項中未列明而稱未知悉。

國立清華大學 113 學年度碩士班考試入學試題

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　　*請在【答案卷】作答

Notation.

- $\mathbb{R}$ = the set of all real numbers;
- $M_{m\times n}(\mathbb{R})$ = the set of all real $m \times n$ matrices;
- $P_n(\mathbb{R})$ = the set of all polynomials of degrees at most n with real coefficients;
- $C^\infty(\mathbb{R})$ = the set of all infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$;
- If f is a differentiable function, we write f' for its derivative;
- If v is a vector in an inner product space, we write $\|v\|$ for its norm.

1. Let $a \in \mathbb{R}$, and let $T\colon P_3(\mathbb{R}) \to P_2(\mathbb{R})$ be the function defined by

$$T\big(f(x)\big) = f'(x-a) + (ax+1)f''(x)$$

for all $f(x) \in P_3(\mathbb{R})$.

(a) **(8 points)** For which real numbers a is the function T linear? Prove your answer.

(b) **(10 points)** Find all real numbers a such that T is not surjective.

2. Let V be an $\mathbb{R}$-vector space, and let $T\colon V \to V$ be a linear operator on V.

(a) **(8 points)** Show that if $V = C^\infty(\mathbb{R})$ and $T(f) = f'$ for all $f \in C^\infty(\mathbb{R})$, then T has infinitely many eigenvalues.

(b) **(10 points)** Give a **detailed** proof that if $V = P_n(\mathbb{R})$, then any linear operator T on V has only finitely many eigenvalues. Your proof should contain enough details so that the grader can see clearly why it does not work for $V = C^\infty(\mathbb{R})$.

3. Let V be a (not necessarily finite-dimensional) real inner product space. Let u_1 and u_2 be two distinct vectors in V, and let

$$S = \big\{v \in V \bigm| \|v-u_1\| = \|v-u_2\|\big\}.$$

(a) **(10 points)** What is the necessary and sufficient condition on u_1 and u_2 so that S is a subspace of V? Prove your answer.

(b) **(10 points)** When V is finite-dimensional and S is a subspace, what is the relation between $\dim V$ and $\dim S$? Prove your answer.

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4. Let $A \in M_{m\times n}(\mathbb{R})$ be a matrix and $b \in \mathbb{R}^m$ be a column vector such that the system of linear equations $Ax = b$ has no solutions for $\|x\| > 2024$, but has at least one solution for $\|x\| \le 2024$. Give a proof or an explicit counterexample for each of the following statements.

   (a) **(10 points)** The system of linear equations $Ax = b$ has only one solution for $x \in \mathbb{R}^n$.

   (b) **(10 points)** $m \ge n$.

5. **(12 points)** Let u_1, u_2, v_1, v_2 be column vectors in $\mathbb{R}^n$ such that u_1, u_2 are linearly independent and v_1, v_2 are linearly independent. Prove that the following two conditions are equivalent.

   (a) There exists an $n \times n$ orthogonal matrix A such that $Au_1 = v_1$ and $Au_2 = v_2$.

   (b) $\|u_1\| = \|v_1\|$, $\|u_2\| = \|v_2\|$, and $\|u_1 - u_2\| = \|v_1 - v_2\|$.

6. **(12 points)** Does there exist a matrix $A \in M_{3\times 3}(\mathbb{R})$ such that

$$A^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}?$$

Prove your answer.