注意：考試開始鈴響前，不得翻閱試題，並不得書寫、畫記、作答。

國立清華大學 112 學年度碩士班考試入學試題

系所班組別：數學系

科目代碼：0102
考試科目：線性代數

一作答注意事項一

1. 請核對答案卷（卡）上之准考證號、科目名稱是否正確。

2. 考試開始後，請於作答前先翻閱整份試題，是否有污損或試題印刷不清，得舉手請監試人員處理，但不得要求解釋題意。

3. 考生限在答案卷上標記「▛ 由此開始作答」區內作答，且不可書寫姓名、准考證號或與作答無關之其他文字或符號。

4. 答案卷用盡不得要求加頁。

5. 答案卷可用任何書寫工具作答，惟為方便閱卷辨識，請儘量使用藍色或黑色書寫；答案卡限用 2B 鉛筆畫記；如畫記不清（含未依範例畫記）致光學閱讀機無法辨識答案者，其後果一律由考生自行負責。

6. 其他應考規則、違規處理及扣分方式，請自行詳閱准考證明上「**國立清華大學試場規則及違規處理辦法**」，無法因本試題封面作答注意事項中未列明而稱未知悉。

(1) (10%) Let

$$A = \begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{pmatrix}.$$

For which triples $C^t = (c_1, c_2, c_3)$ does the system $AX = C$ have a solution? And find the solutions, if any. Here C^t is the transpose of C.

(2) Let

$$A = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

   (a) (5%) Prove that the left-multiplication transformation L_A is a reflection.

   (b) (10%) Find the axis in R^2 about which L_A reflects.

   (c) (5%) Prove that L_{AB} and L_{BA} are rotations.

(3) (20%) Let $V = P_2(R)$ be the space of all polynomials with coefficients in R, having degree at most 2. Define a linear operator T on V by

$$T(f(x)) = -xf''(x) + f'(x) + 2f(x).$$

Find the minimal polynomial of T.

(4) (20%) Describe all linear operators T on R^2 such that T is diagonalizable and $T^3 - 2T^2 + T = T_0$, where T_0 is the zero transformation.

(5) (15%) Let g be a non-degenerate form on a finite-dimensional space V. Show that each linear operator T has an operator T' such that

$$g(Tv, w) = g(v, T'w)$$

for all v, w.

(6) (a) (5%) If N is a nilpotent 3×3 matrix over C, prove that $A = I + \frac{1}{2}N - \frac{1}{8}N^2$ satisfies $A^2 = I + N$, i.e., A is a square root of $I + N$.

   (b) (10%) If N is a nilpotent $n \times n$ matrix over C, find a square root of $I + N$.