注意：考試開始鈴響前，不得翻閱試題，

並不得書寫、畫記、作答。

國立清華大學 112 學年度碩士班考試入學試題

系所班組別：數學系

科 目 代 碼 ：0101

考 試 科 目 ：高等微積分

—作答注意事項—

1. 請核對答案卷（卡）上之准考證號、科目名稱是否正確。
2. 考試開始後，請於作答前先翻閱整份試題，是否有污損或試題印刷不清，得舉手請監試人員處理，但不得要求解釋題意。
3. 考生限在答案卷上標記「➡ 由此開始作答」區內作答，且不可書寫姓名、准考證號或與作答無關之其他文字或符號。
4. 答案卷用盡不得要求加頁。
5. 答案卷可用任何書寫工具作答，惟為方便閱卷辨識，請儘量使用藍色或黑色書寫；答案卡限用 2B 鉛筆畫記；如畫記不清（含未依範例畫記）致光學閱讀機無法辨識答案者，其後果一律由考生自行負責。
6. 其他應考規則、違規處理及扣分方式，請自行詳閱准考證明上「**國立清華大學試場規則及違規處理辦法**」，無法因本試題封面作答注意事項中未列明而稱未知悉。

國立清華大學 112 學年度碩士班考試入學試題

系所班組別：數學系碩士班

考試科目 (代碼)：高等微積分 (0101)

1.(10 pts) Let A be a nonempty set of real numbers which is bounded below. Let $-A = \{-x | \, x \in A\}$. Prove that $\inf A = -\sup(-A)$.

2. (15 pts) Let E be a nonempty subset of $\mathbb{R}^n$. Let E' be the set of all limit points (accumulation point). (i) Is E' a closed set ? (ii) Does E and E' always have the same limit points? Prove the statements or give counterexamples for (i) and (ii). (iii) If $E = \{(x, \sin \frac{1}{x}) | \, x \in (0, 1)\}$. What is E' ?

3. (10 pts) Let a_1 and $a_{n+1} = 1 + \sqrt{a_n}$, $n \in \mathbb{N}$. Find

$$\limsup_{n \to \infty} a_n.$$

4. (15 pts) (i) Suppose $a_n \geq 0$, $n \in \mathbb{N}$ and $\sum a_n$ diverges. Prove $\sum a_n/(1 + a_n)$ also diverges. (ii) Suppose $\sum a_n$ is a series of real numbers which converges absolutely. Prove that every rearrangement of $\sum a_n$ converges to the same sum.

5. (10 pts) Use the finite open covering property of the compactness to show the following. If f is a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly continuous on X.

6.(15 pts) Give the reasons in your computation. (i) Let $a > 0$. Find

$$\lim_{n\to\infty} \int_a^\infty \frac{\sin(nx)}{nx} dx.$$

(ii) What is the answer for (i) if $a = 0$?

(iii) Let $[x]$ be the greatest integer function. That is $[x] = \sup\{n \mid n \le x,\ n \in \mathbb{Z}\}$. Find

$$\int_0^2 x d[x].$$

7. (15 pts) Let $f : \mathbb{R}^5 \to \mathbb{R}^2$ defined by $f = (f_1, f_2)$ with

$$f_1(x_1, x_2, y_1, y_2, y_3) = 2e^{x_1} + x_2 y_1 - 4y_2 + 3$$
$$f_2(x_1, x_2, y_1, y_2, y_3) = x_2 \cos x_1 - 6x_1 + 2y_1 - y_3.$$

Note that $f(0, 1, 3, 2, 7) = (0, 0)$. (i) Prove that there exists a function $g : \mathbb{R}^3 \to \mathbb{R}^2$ defined on a neighborhood of $(3, 2, 7)$ so that $f(g(y_1, y_2, y_3), y_1, y_2, y_3) = 0$. (ii) Find the derivative of g at $(3, 2, 7)$.

8.(10 pts) (i) Let $(x, y) \in \mathbb{R}^2 \setminus (0, 0)$, $P(x, y) = \frac{-y}{x^2+y^2}$ and $Q(x, y) = \frac{x}{x^2+y^2}$. Let C be the curve goes from $(a, 0), a > 0$ to itself once along $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, b > 0$. Find

$$\oint_C P(x, y)dx + Q(x, y)dy.$$

(ii) Can we find a function $f : \mathbb{R}^2 \setminus (0, 0) \to \mathbb{R}$ such that

$$\frac{\partial f}{\partial x} = P(x, y) \text{ and } \frac{\partial f}{\partial y} = Q(x, y)?$$

Give the reason for your answer.