注意：考試開始鈴響前，不得翻閱試題，
並不得書寫、畫記、作答。

國立清華大學 111 學年度碩士班考試入學試題

系所班組別：數學系

科 目 代 碼 ：0102

考 試 科 目 ：線性代數

—作答注意事項—

1. 請核對答案卷（卡）上之准考證號、科目名稱是否正確。
2. 考試開始後，請於作答前先翻閱整份試題，是否有污損或試題印刷不清，得舉手請監試人員處理，但不得要求解釋題意。
3. 考生限在答案卷上標記「由此開始作答」區內作答，且不可書寫姓名、准考證號或與作答無關之其他文字或符號。
4. 答案卷用盡不得要求加頁。
5. 答案卷可用任何書寫工具作答，惟為方便閱卷辨識，請儘量使用藍色或黑色書寫；答案卡限用 2B 鉛筆畫記；如畫記不清（含未依範例畫記）致光學閱讀機無法辨識答案者，其後果一律由考生自行負責。
6. 其他應考規則、違規處理及扣分方式，請自行詳閱准考證明上「**國立清華大學試場規則及違規處理辦法**」，無法因本試題封面作答注意事項中未列明而稱未知悉。

國立清華大學 111 學年度碩士班考試入學試題

系所班組別：數學系碩士班

考試科目（代碼）：線性代數（0102）

 *請在【答案卷、卡】作答

1 (10%) Let V be a finite dimensional vector space over $\mathbb{R}$, and v_1, v_2 be two distinct vectors in V. Show that there is an $\mathbb{R}$-linear transofrmation $f : V \to \mathbb{R}$ for which

$$f(v_1) \neq f(v_2).$$

2 (18%) Let $V := \mathrm{Mat}_{1\times 3}(\mathbb{R})$ and define $f : V \to \mathbb{R}$ by

$$f(x, y, z) := \det \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 2 \\ x & y & z \end{pmatrix}.$$

(i) Show that f is a linear transformation over $\mathbb{R}$.

(ii) Put $W := \mathrm{Ker} f$. Find an $\mathbb{R}$-basis of W.

(iii) Let V/W be the quotient space of V by W, and elements in V/W are denoted by $\bar{v}$ for $v \in V$. Show that the map $\bar{f} : V/W \to \mathbb{R}$ given by

$$\bar{f}\left(\overline{(a, b, c)}\right) := \det \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 2 \\ a & b & c \end{pmatrix}$$

is well-defined and is an isomorphism of vector spaces.

3 (14%) Find the Journal canonical form of the following matrix

$$\begin{pmatrix} 2 & -4 & 2 & 2 \\ -2 & 0 & 1 & 3 \\ -2 & -2 & 3 & 3 \\ -2 & -6 & 3 & 7 \end{pmatrix}.$$

4 (12%) Suppose that V is finite dimensional inner product space over $\mathbb{C}$, and T is a normal linear operator on V such that $T^9 = T^8$. Prove that T is self-adjoint and $T^2 = T$.

5 (20%) Let V be a finite dimensional vector space over $\mathbb{C}$ with two inner products $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle'$. Prove the following.

(i) There exists a unique linear operator T on V so that $\langle x, y \rangle' = \langle T(x), y \rangle$ for all $x, y \in V$.

(ii) The linear operator T in (i) is positive definite with respect to both inner products.

6 (14%) Let $V := \mathbb{R}^n$ and let $W \subset V$ be the vector subspace defined as the set of solutions of $x_1 + \cdots + x_n = 0$. Define $W^0 := \{f \in V^* | f(w) = 0 \text{ for all } w \in W\}$, where

$$V^* := \{f : V \to \mathbb{R} | f \text{ is a linear transformation over } \mathbb{R}\},$$

the dual space of V. Show that W^0 is equal to the set of all f of the form

$$f\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \lambda(a_1 + \cdots + a_n) \text{ for some } \lambda \in \mathbb{R}.$$

7. (12%) Given a nonzero matrix $A \in \mathrm{Mat}_n(\mathbb{R})$ and a nonzero vector $\mathbf{b} \in \mathrm{Mat}_{n\times 1}(\mathbb{R})$, show that if there exists a row vector $C \in \mathrm{Mat}_{1\times n}(\mathbb{R})$ for which $CA = 0$ and $C\mathbf{b} = 1$, then $A\mathbf{x} = \mathbf{b}$ has no solution.