注意：考試開始鈴響前，不得翻閱試題，

並不得書寫、畫記、作答。

國立清華大學 110 學年度碩士班考試入學試題

系所班組別：數學系

科 目 代 碼 ：0102

考 試 科 目 ：線性代數

一作答注意事項一

1. 請核對答案卷（卡）上之准考證號、科目名稱是否正確。
2. 考試開始後，請於作答前先翻閱整份試題，是否有污損或試題印刷不清，得舉手請監試人員處理，但不得要求解釋題意。
3. 考生限在答案卷上標記「由此開始作答」區內作答，且不可書寫姓名、准考證號或與作答無關之其他文字或符號。
4. 答案卷用盡不得要求加頁。
5. 答案卷可用任何書寫工具作答，惟為方便閱卷辨識，請儘量使用藍色或黑色書寫；答案卡限用 2B 鉛筆畫記；如畫記不清（含未依範例畫記）致光學閱讀機無法辨識答案者，其後果一律由考生自行負責。
6. 其他應考規則、違規處理及扣分方式，請自行詳閱准考證明上「**國立清華大學試場規則及違規處理辦法**」，無法因本試題封面作答注意事項中未列明而稱未知悉。

國立清華大學 110 學年度碩士班考試入學試題

系所班組別：數學系碩士班

考試科目（代碼）：線性代數（0102）

共__1__頁，第__1__頁　　*請在【答案卷、卡】作答

Notation: $\mathbb{R}$ denotes the field of real numbers; $\mathbb{C}$ denotes the field of complex numbers. F denotes an arbitrary field; $M_{m\times n}(F)$ denotes the set of all $m \times n$ matrices with entries in F. If T is a linear transformation, $R(T)$ denotes the range of T, and $N(T)$ denotes the null space of T. If $A \in M_{m\times n}(F)$, A^t denotes the transpose of A, and L_A denotes the linear transformation from F^n to F^m that sends each vector $v \in F^n$ to $Av \in F^m$.

1. **(12 points)** Let V and W be F-vector spaces, and let $T\colon V \to W$ be a linear transformation. Prove that $\dim R(T)+\dim N(T) = \dim V$ if V is finite-dimensional.

2. **(10 points)** Find a matrix $A \in M_{3\times 3}(\mathbb{R})$ such that

$$R(L_A) = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; 3a - 2b + 4c = 0 \right\} \text{ and } N(L_A) = \left\{ \begin{pmatrix} 2t \\ 3t \\ -t \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; t \in \mathbb{R} \right\}.$$

   You need to show that the matrix you find has the required properties.

3. **(12 points)** Let $A \in M_{m\times n}(F)$. Show that the system of linear equations $Ax = b$ has a solution for all $b \in F^m$ if and only if the system of linear equations $A^t x = 0$ has no nonzero solutions.

4. **(12 points)** Let $A \in M_{m\times n}(F)$ and $B \in M_{n\times m}(F)$. Show that if $\operatorname{rank}(AB) = m$, then $\operatorname{rank}(BA) = m$.

5. Let $T\colon V \to V$ be a linear operator on a finite-dimensional F-vector space V.

   (a) **(6 points)** State the definition of eigenvectors of T.

   (b) **(6 points)** Give an explicit example of T that has no eigenvectors.

   (c) **(8 points)** Prove that T has an eigenvector if $F = \mathbb{C}$.

6. **(10 points)** Let $A \in M_{n\times n}(F)$. Show that if $Q \in M_{n\times n}(F)$ is an invertible matrix such that $Q^{-1}AQ$ is diagonal, then each column vector of Q is an eigenvector of L_A.

7. **(12 points)** Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Show that if T preserves the Euclidean distance between any two points, that is, $\|T(u) - T(v)\| = \|u - v\|$ for any $u, v \in \mathbb{R}^n$, then the matrix representation of T relative to the standard basis is an orthogonal matrix.

8. **(12 points)** Let $A \in M_{n\times n}(\mathbb{R})$ be a real symmetric matrix. Show that there exists a real symmetric matrix B such that $B^3 = A$.