注意：考試開始鈴響前，不得翻閱試題，
並不得書寫、畫記、作答。

國立清華大學 115 學年度碩士班考試入學試題

系所班組別：計算與建模科學研究所

科 目 代 碼 ：0301

考 試 科 目 ：數學分析

一作答注意事項一

1. 請核對答案卷（卡）上之准考證號、科目名稱是否正確。
2. 考試開始後，請於作答前先翻閱整份試題，是否有污損或試題印刷不清，得舉手請監試人員處理，但不得要求解釋題意。
3. 考生限在答案卷上標記「⬇由此開始作答」區內作答，且不可書寫姓名、准考證號或與作答無關之其他文字或符號。
4. 答案卷用盡不得要求加頁。
5. 答案卷可用任何書寫工具作答，惟為方便閱卷辨識，請儘量使用藍色或黑色書寫；答案卡限用 2B 鉛筆畫記；如畫記不清（含未依範例畫記）致光學閱讀機無法辨識答案者，其後果一律由考生自行負責。
6. 提早交卷者不可攜出試題，請將試題連同答案卷(卡)交予監考人員，攜出將以違規論處。當節考試結束後試題將置於講台，可自行取回，本試場不負保管責任。
7. 其他應考規則、違規處理及扣分方式，請自行詳閱准考證明上「**國立清華大學試場規則及違規處理辦法**」，無法因本試題封面作答注意事項中未列明而稱未知悉。

1. Evaluate the following integrals:
   (a) $\int \sin(\ln(x))\,dx$. **(6 points)**
   (b) $\int_0^1 \int_x^1 \exp\left(\frac{x}{y}\right) dy\, dx$. **(7 points)**
   (c) $\iint_R \sin(9x^2+4y^2)dA$ where R is the region in the first quadrant bounded by the ellipse $9x^2+4y^2=1$. **(7 points)**

2. Let $f(x,y)=\begin{cases} 0 & \text{if } y \le 0 \text{ or } y \ge x^4 \\ 1 & \text{if } 0 < y < x^4 \end{cases}$
   (a) Show that $f(x,y) \to (0,0)$ as $(x,y) \to (0,0)$ along any path through $(0,0)$ of the form $y=mx^a$ where m and a are constant and $a<4$. **(10 points)**
   (b) Despite part (a), show that f is discontinuous at (0,0). **(10 points)**

3. If $a>0$, find the area of the surface generated by rotating the loop of the curve $3ay^2=x(a-x)^2$ about the x-axis. **(10 points)**

4. Find the absolute maximum and minimum values of $f(x,y)=xy^2$ on the set $\mathrm{D}=\{(x,y):\ x\ge 0, y\ge 0, x^2+y^2\le 3\}$. **(10 points)**

5. Determine whether the following series is convergent, or divergent.
   (a) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$. **(7 points)**
   (b) $\sum_{n=1}^{\infty} a_n$ where $a_{n+1}=\frac{2+\cos(n)}{\sqrt{n}}a_n$, and $a_1=1$. **(8 points)**

6. An $n\times n$ matrix L is called a **unit lower triangular** matrix if it is a lower triangular matrix with 1's on the main diagonal.
   (i) Let A be an $n\times n$ matrix admitting a decomposition $A=LU$, where L is a **unit lower triangular** matrix and U is an invertible upper triangular matrix. Prove that this decomposition is unique. **(5 points)**
   (ii) Find the 4×4 **unit lower triangular** matrix L and the upper triangular matrix U such that

$$LU=\begin{bmatrix} 2 & -1 & 0 & 2 \\ 4 & 1 & 2 & 2 \\ -2 & 5 & -1 & 3 \\ 6 & -1 & 2 & 4 \end{bmatrix}.\ \textbf{(7 points)}$$

7. Let $V = P_3(\mathbb{R})$ be the vector space of all polynomials with real coefficients of degree at most 3. Consider a linear operator $T : V \to V$ defined by

$$T(f(x)) = 2f'''(0)x^3 + (6f''(0) + kf'''(0))x^2,$$

where $' := \frac{\mathrm{d}}{\mathrm{d}x}$ and $k \in \mathbb{R}$ is a real constant.

(i) Does the kernel of T depend on the value of k? Justify your answer. **(5 points)**

(ii) Does the characteristic polynomial of T depend on the value of k? Justify your answer. **(5 points)**

(iii) Find all values of k such that T is diagonalizable. **(8 points)**

8. Consider a matrix A_n $(n \geq 3)$ whose entry in the i-th row and j-th column is $\frac{1}{i+j-1}$, where $1 \leq i, j \leq n$. Let M^T denote the transpose of the matrix M. Show that:

(i) $PA_3P^T = 2026$, where $P = [-28 \quad 42 \quad 90]$. **(5 points)**

(ii) $XA_nX^T > 0$ for any non-zero matrix $X \in \mathbb{R}^{1\times n}$. **(12 points)**

(You can get 5 points for a complete proof of the case $n = 3$.)

(iii) There exists an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ such that QA_nQ^T is a diagonal matrix. **(8 points)**

9. Let $A = \operatorname{diag}(M_1, M_2, \cdots, M_{500}, N_1, N_2, \cdots, N_{13})$ be a 2026×2026 matrix consisting of 500 blocks of 4×4 matrices M_j and 13 blocks of 2×2 matrices N_k, where

$$M_j = \begin{bmatrix} 10j+5 & 1 & 0 & 0 \\ 0 & 10j+5 & 0 & 0 \\ 0 & 0 & 10j+1 & 1 \\ 0 & 0 & -2 & 10j+4 \end{bmatrix} \text{ and } N_k = \begin{bmatrix} 10k+7 & 1 \\ 0 & 10k+7 \end{bmatrix}.$$

Consider the linear operator $T : \mathbb{R}^{2026} \longrightarrow \mathbb{R}^{2026}$ defined by $T(v) = Av$.

(i) Decompose $\mathbb{R}^{2026}$ into a direct sum of T-invariant subspaces $\mathbb{R}^{2026} = W_1 \oplus W_2 \oplus \cdots \oplus W_n$ such that each W_i cannot be further decomposed into a direct sum of smaller T-invariant subspaces. Justify the value of n and provide a basis for each W_i. **(12 points)**

(ii) Find the total number of T-invariant subspaces of $\mathbb{R}^{2026}$. (Prove your answer.) **(8 points)**