注意：考試開始鈴響前，不得翻閱試題，
並不得書寫、畫記、作答。

國立清華大學 114 學年度碩士班考試入學試題

系所班組別：計算與建模科學研究所

科 目 代 碼 ：0301

考 試 科 目 ：數學分析

—作答注意事項—

1. 請核對答案卷（卡）上之准考證號、科目名稱是否正確。
2. 考試開始後，請於作答前先翻閱整份試題，是否有污損或試題印刷不清，得舉手請監試人員處理，但不得要求解釋題意。
3. 考生限在答案卷上標記「⬇ 由此開始作答」區內作答，且不可書寫姓名、准考證號或與作答無關之其他文字或符號。
4. 答案卷用盡不得要求加頁。
5. 答案卷可用任何書寫工具作答，惟為方便閱卷辨識，請儘量使用藍色或黑色書寫；答案卡限用 2B 鉛筆畫記；如畫記不清（含未依範例畫記）致光學閱讀機無法辨識答案者，其後果一律由考生自行負責。
6. 其他應考規則、違規處理及扣分方式，請自行詳閱准考證明上「**國立清華大學試場規則及違規處理辦法**」，無法因本試題封面作答注意事項中未列明而稱未知悉。

國立清華大學 114 學年度碩士班考試入學試題

系所班組別：計算與建模科學研究所

考試科目（代碼）：數學分析（0301）

　　*請在【答案卷】作答

1. (10 points) Let $x_1, x_2, ..., x_9$ be real numbers satisfying $x_1 + 2x_2 + \cdots + 9x_9 = 1$. Prove that
$$x_1^2 + \frac{x_2^2}{2} + \cdots + \frac{x_9^2}{9} \geq \frac{1}{2025}.$$
Furthermore, determine all $x_1, ..., x_9$ for which equality holds.

2. (15 points) Let $f(x,y) = \begin{cases} \dfrac{x^3}{2x^2+3y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$
Prove that f has all directional derivatives at $(0,0)$, but f is not differentiable at $(0,0)$.

3. (15 points) Let $F(x) = \dfrac{1 - \sin\frac{\pi x}{2}}{(1-x)^2}$ for $x \in (0,1)$.
   (i) Evaluate the limit $\lim\limits_{x \uparrow 1} F(x)$.
   (ii) Evaluate the integral $\displaystyle\int_0^1 \int_0^z \int_0^y F(x)\, \mathrm{d}x\mathrm{d}y\mathrm{d}z$.

4. (15 points) On $\mathbb{R}^2$, Γ is a **cardioid** (heart-shaped curve) given parametrically by
$$x(t) = 4\cos t \sin^2\frac{t}{2},\ \ y(t) = 2\sin t - \sin 2t,\ \ t \in \mathbb{R}.$$
   (i) Prove that its equation in polar coordinates can be expressed as $r = 4\sin^2\frac{\theta}{2}$.
   (ii) Find the area of the region enclosed by Γ.
   (iii) Find the perimeter (周長) of Γ.

5. (20 points) For $n \in \mathbb{N}$, let $P_n(x) = \displaystyle\sum_{k=1}^{n} (-1)^k \mathrm{C}_k^n (x-k)^n$, $x \in \mathbb{R}$, where $\mathrm{C}_k^n := \dfrac{n!}{k!(n-k)!}$ is the binomial coefficient $(1 \leq k \leq n)$.

   (i) Evaluate the integral $\displaystyle\int_0^\infty \frac{P_3''(x) - 2P_2'(x) - 2P_1(x)}{3 - P_2(x)}\,\mathrm{d}x$.
   (ii) Prove that
$$\frac{P_n'(2)}{P_{n-1}(2) - P_{n-1}(1) - 1} = n,\ \forall\, n \geq 2.$$

6. (7 points) Find an invertible 2×2 matrix P for which $PAP^{-1}=B$ where

$$A=\begin{pmatrix}1 & 2025\\ 0 & 1\end{pmatrix} \text{ and } B=\begin{pmatrix}1 & 99\\ 0 & 1\end{pmatrix}.$$

7. Let $v=\begin{pmatrix}2\\1\\1\end{pmatrix}$, $w_1=\begin{pmatrix}3\\0\\4\end{pmatrix}$ and $w_2=\begin{pmatrix}0\\2\\0\end{pmatrix}$ be the vector in $\mathbb{R}^3$.

   (i) (7 points) Find the orthogonal projection vector of v onto w_1.

   (ii) (7 points) Find the orthogonal projection of vector of v onto the subspace S of $\mathbb{R}^3$ spanned by the vectors w_1 and w_2.

8. Find the reduced row-echelon form (5 points), rank (2 points), nullspace (5 points) and nullity (2 points) of the matrix $A=\begin{pmatrix}2 & -3 & -6 & -4\\ 1 & 5 & -3 & 11\\ 2 & 7 & -6 & 16\end{pmatrix}$.

9. (12 points) Prove that two finite-dimensional vector spaces V and W are isomorphic if and only if they are of the same dimension.

10. Let $u=\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$ and $v=\begin{pmatrix}1\\-1\\0\\2\end{pmatrix}$ be two vectors in $\mathbb{R}^4$. We define $A=uv^T$.

   (i) (7 points) Compute the determinant of A^{100}.

   (ii) (7 points) Compute the trace of A and find all eigenvalues of A.

   (iii) (7 points) Compute the trace of A^4 and find all eigenvalues of A^4.

   (iv) (7 points) Compute the determinant of $A+I_4$, where I_4 is the 4×4 identity matrix.