注意：考試開始鈴響前，不得翻閱試題，
並不得書寫、畫記、作答。

國立清華大學 113 學年度碩士班考試入學試題

系所班組別：計算與建模科學研究所

科 目 代 碼 ：0301

考 試 科 目 ：數學分析

一作答注意事項一

1. 請核對答案卷（卡）上之准考證號、科目名稱是否正確。
2. 考試開始後，請於作答前先翻閱整份試題，是否有污損或試題印刷不清，得舉手請監試人員處理，但不得要求解釋題意。
3. 考生限在答案卷上標記「⬇由此開始作答」區內作答，且不可書寫姓名、准考證號或與作答無關之其他文字或符號。
4. 答案卷用盡不得要求加頁。
5. 答案卷可用任何書寫工具作答，惟為方便閱卷辨識，請儘量使用藍色或黑色書寫；答案卡限用 2B 鉛筆畫記；如畫記不清（含未依範例畫記）致光學閱讀機無法辨識答案者，其後果一律由考生自行負責。
6. 其他應考規則、違規處理及扣分方式，請自行詳閱准考證明上「國立清華大學試場規則及違規處理辦法」，無法因本試題封面作答注意事項中未列明而稱未知悉。

國立清華大學 113 學年度碩士班考試入學試題

系所班組別：計算與建模科學研究所

考試科目（代碼）：數學分析 (0301)

 *請在【答案卷、卡】作答

1. Find the derivative for each of the following functions,
   (i) (6 points) Find f_{xxy} for $f(x,y) = y^2\ln(x)$.
   (ii) (6 points) Find $\frac{\partial^3 f}{\partial y \partial x^2}$ for $f(x,y) = e^{xy}$.

2. (i) (6 points) Find the largest and smallest values of $f(x) = 2x^3 + 3x^2 - 12x$ on $[-4,3]$.
   (ii) (6 points) Use the method of Lagrange multipliers to find the maximum value of $f(x,y) = xy$ subject to the constraint $x^2 + 4y^2 = 2$.

3. (6 points) Find a function f such that $f'(x) = x^3$ and the line $x + y = 0$ is tangent to the graph of f.

4. (7 points) Find the directional derivative of the function $f(x,y,z) = \sin(yz) + \ln(x^2)$ at $(1,1,\pi)$ in the direction of the vector $\vec{v} = \langle 1,1,-1\rangle$.

5. (i) (7 points) Compute the sixth order McLaurin polynomial of $f(x) = \sqrt{1-x^2}$.
   (ii) (7 points) Compute the second order Taylor polynomial for $f(x) = x^{\log(x)}$ about the number 1.

6. (8 points) Find the highest and lowest points on the curve $x^2 + xy + y^2 = 12$.

7. (i) (8 points) Determine $\left(\int_0^1 \frac{dx}{\sqrt{1-x^4}}\right) \div \left(\int_0^1 \frac{dx}{\sqrt{1+x^4}}\right)$.

   (ii) (8 points) Determine $\int_0^2 \frac{e^x}{e^{1-x}+e^{x-1}}\,dx$.

8. Let $T: P_4(\mathbb{R}) \to P_4(\mathbb{R})$ be the linear operator with

$$T(1) = -1, T(x) = 2x, T(x^2) = 2 - x^3 - x^4,$$

$$T(x^3) = -2 + x^2 + 2x^3 + x^4, T(x^4) = 1 + x^2 + 3x^4.$$

   (i) (6 points) Discuss the iteration $T^n(2024 + 2024x^2 + 2024x^3)$ for each $n \in \mathbb{N}$ and justify your answer carefully.

   (ii) (8 points) Determine the characteristic polynomial of T.

   (iii) (10 points) Find the Jordan canonical form of T.

9. (10 points) Let P_i be invertible 6×6 matrices for $i=1,2,3$ and define the matrix A by

$$A=\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 66 & 6 & 3 & 3^2 & 3^3 & 3^4 \\ 66 & 6 & 1 & 1^2 & 1^3 & 1^4 \\ 66 & 6 & 0 & 0 & 0 & 0 \\ 66 & 6 & 2 & 2^2 & 2^3 & 2^4 \\ 66 & 6 & 4 & 4^2 & 4^3 & 4^4 \end{pmatrix}.$$

Find $\det(P_3^{-1}P_2^{-1}P_1^{-1}AP_1P_2P_3)$.

10. Define the matrix A by

$$A=\begin{pmatrix} 1 & 2 & 3 \\ -1 & 6 & 1 \\ 0 & -2 & 5 \end{pmatrix}$$

(i) (8 points) Find an LU decomposition of A. That is, $A=LU$ where L is a lower triangular matrix and U is an upper triangular matrix.

(ii)(8 points) Find a QR decomposition of A. In other words, A can be represented as$A=QR$ where $Q=[q_1\, q_2\, q_3]$ with orthonormal columns q_1, q_2, q_3 and R is an 3×3 invertible upper triangular matrix.

11. Let $v\in\mathbb{C}^{m\times 1}$ be given and consider the linear system $Ax=v$ with A is a $m\times n$ matrix over $\mathbb{C}$ and $ker(A)=\{x\in\mathbb{C}^{n\times 1} \mid Ax=0\}=\{0\}$.

(i) (7 points) Prove that $ker(A^HA)=ker(PA)$ where P is an invertible $m\times m$ matrix over $\mathbb{C}$.

(ii) (8 points) If $m=n$ and denote the solution of the linear system

$$A^HAx=A^Hv$$

and the solution of $Ax=v$ are $\overline{x}$ and $\widehat{x}$, respectively. Show that $\overline{x}=\widehat{x}$.

(iii) (10 points) Prove that

$$\|v-Ax\|\geq\|v-A\overline{x}\|$$

for all $x\in\mathbb{C}^{n\times 1}$.