注意：考試開始鈴響前，不得翻閱試題，並不得書寫、畫記、作答。

國立清華大學 110 學年度碩士班考試入學試題

系所班組別：計算與建模科學研究所

科目代碼：0301

考試科目：數學分析

—作答注意事項—

1. 請核對答案卷（卡）上之准考證號、科目名稱是否正確。
2. 考試開始後，請於作答前先翻閱整份試題，是否有污損或試題印刷不清，得舉手請監試人員處理，但不得要求解釋題意。
3. 考生限在答案卷上標記「由此開始作答」區內作答，且不可書寫姓名、准考證號或與作答無關之其他文字或符號。
4. 答案卷用盡不得要求加頁。
5. 答案卷可用任何書寫工具作答，惟為方便閱卷辨識，請儘量使用藍色或黑色書寫；答案卡限用 2B 鉛筆畫記；如畫記不清（含未依範例畫記）致光學閱讀機無法辨識答案者，其後果一律由考生自行負責。
6. 其他應考規則、違規處理及扣分方式，請自行詳閱准考證明上「**國立清華大學試場規則及違規處理辦法**」，無法因本試題封面作答注意事項中未列明而稱未知悉。

1. Calculate the limit of the following x_n with respect to $n \to \infty$:

(i) (7 分) $x_n = \sum_{k=1}^{n} \frac{1}{1^3 + \cdots + k^3}$

(ii) (10 分) $x_n = \frac{e^{n^{-2}} - \sec^2(n^{-1})}{\tan(n^{-\beta})}$ with $\beta > 0$.

P.S. For (i) you can use the fact $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

2. **(8 分)** Calculate the volume of the solid formed by revolving the region bounded by the graphs of $y = x^3 + x + 1, y = 1$ and $x = 1$ about the line $x = 2$.

3. **(8 分)** Let $T(x, y, z) = 20 + 2x + 2y + z^2$ represent the temperature at each point on the sphere $x^2 + y^2 + z^2 = 11$. Calculate the minimum and maximum temperatures on the curve formed by the intersection of the plane $x + y + z = 3$ and this sphere.

4. **(8 分)** Let D be the region bounded by the lines

$$x - 2y = 0, x - 2y + 4 = 0, x + y - 4 = 0, x + y - 1 = 0.$$

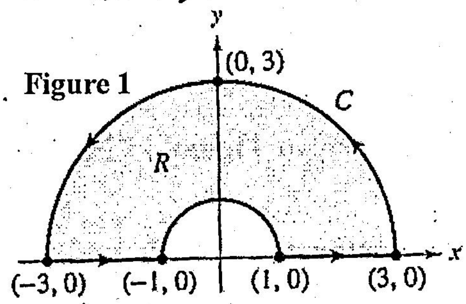
Calculate the integral $\iint_D 3xy\,dx\,dy$.

5. **(8 分)** Calculate

$$\oint_C (\arctan x + y^2)dx + (e^y - x^2)dy,$$

where C is the path enclosing the annular region R shown in **Figure 1**.

Figure 1



6. Let $f(x)=\sqrt{1+x},\ x\in(-1,\infty)$.

(i) (10 分) Show that the Taylor series of $f(x)$ at $x=0$ is

$$f_{\mathrm{TL}}(x)=1-\sum_{k=0}^{\infty}\frac{2C_k^{2k}}{k+1}\left(-\frac{x}{4}\right)^{k+1};\qquad x\in(-1,\infty).$$

(ii) (8 分) By integrating $f(-x^2)$ over a suitable region and using the formula of f_{TL}, verify the value of

$$\sum_{k=0}^{\infty}\left(\frac{1}{2k+2}-\frac{1}{2k+3}\right)\frac{C_k^{2k}}{4^k}.$$

Here you need only calculate it without the rigorous analysis.

(iii) (8 分) Let $I=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$ and $A=\begin{bmatrix}0&20&21&20\\0&0&20&21\\0&0&0&20\\0&0&0&0\end{bmatrix}$. Find a matrix B such that $B^{16}=I+A$.

Hint: (i) is a key for obtaining a B, where you shall notice $16=2^4$.

7. (8 分) Show that the area of a triangle with the vertices (x_1,y_1), (x_2,y_2), and (x_3,y_3) is

$$\text{Area}=\frac{1}{2}\begin{vmatrix}x_1&y_1&1\\x_2&y_2&1\\x_3&y_3&1\end{vmatrix},$$

where $0<x_1<x_3<x_2$ and $0<y_1<y_2<y_3$.

8. Let V be an inner product space. Prove that

(i) (8 分) If $S=\{v_1,v_2,\dots,v_n\}$ is an orthogonal set of nonzero vectors in V, then S is linearly independent.

(ii) (8 分) If $B = \{u_1, u_2, \dots, u_n\}$ is an orthonormal basis for V, then the coordinate representation of a vector w relative to B is $w = \langle w, u_1\rangle u_1 + \langle w, u_2\rangle u_2 + \cdots + \langle w, u_n\rangle u_n$.

9. Let $A = \begin{bmatrix} 10 & 18 \\ -6 & -11 \end{bmatrix}$.

(i) (8 分) Find the eigenvalues and corresponding eigenvectors of A.

(ii) (8 分) Find A^8.

10. Define the spectral radius of an $n \times n$ matrix M as

$$\rho(M) = \max_{1 \le i \le n}\{|\lambda_1|, \dots, |\lambda_n|\},$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of M. Let A and B be two square matrices. Prove or disprove each of the following

(i) (8 分) $\rho(AB) \le \rho(A)\rho(B)$.

(ii) (8 分) $\rho(A+B) \le \rho(A) + \rho(B)$.

11. Let V be an inner product space and $u, v \in V$.

(i) (10 分) Prove the Cauchy-Schwarz Inequality

$$|\langle u, v\rangle| \le \|u\| \cdot \|v\|.$$

(ii) (9 分) If v is not a zero vector, show that $\|u+v\| = \|u\| + \|v\|$ if and only if there exists a scalar $\sigma \in [0, \infty)$ such that $u = \sigma v$.