注意：考試開始鈴響前，不得翻閱試題，
並不得書寫、畫記、作答。

國立清華大學 109 學年度碩士班考試入學試題

系所班組別：計算與建模科學研究所

科 目 代 碼 ：0301

考 試 科 目 ：數學分析

—作答注意事項—

1. 請核對答案卷（卡）上之准考證號、科目名稱是否正確。
2. 作答中如有發現試題印刷不清，得舉手請監試人員處理，但不得要求解釋題意。
3. 考生限在答案卷上標記「由此開始作答」區內作答，且不可書寫姓名、准考證號或與作答無關之其他文字或符號。
4. 答案卷用盡不得要求加頁。
5. 答案卷可用任何書寫工具作答，惟為方便閱卷辨識，請儘量使用藍色或黑色書寫；答案卡限用 2B 鉛筆畫記；如畫記不清（含未依範例畫記）致光學閱讀機無法辨識答案者，其後果一律由考生自行負責。
6. 其他應考規則、違規處理及扣分方式，請自行詳閱准考證明上「**國立清華大學試場規則及違規處理辦法**」，無法因本試題封面作答注意事項中未列明而稱未知悉。

國立清華大學 109 學年度碩士班考試入學試題

系所班組別：計算與建模科學研究所

考試科目（代碼）：數學分析 (0301)

　　*請在【答案卷、卡】作答

1. Let $f:(-1,1) \to \mathrm{R}$ be a differentiable function satisfying

$$\int_0^x (f(t))^2 \mathrm{d}t = f(x) - x.$$

(i) **(6 分)** Calculate the value of $\frac{\mathrm{d}^5 f}{\mathrm{d}x^5}(0)$.

(ii) **(6 分)** Show that $\int_0^x \left[f(t) - t\big(f(x-t)\big)^2 \right] \mathrm{d}t = \frac{x^2}{2}$.

(iii) **(8 分)** Show that $f(x) > 0$ and

$$\frac{(f(x))^{1010}}{\int_0^x (f(t))^{1010} \mathrm{d}t} \geq 2020 \text{ for } x \in (0,1).$$

2. Suppose that f is a continuous function on R.

(i) **(7 分)** Prove that

$$\int_a^b \frac{f(a+x)+f(b+x)}{f(2a+b-x)+f(a+x)+f(b+x)+f(a+2b-x)} dx = \frac{b-a}{2}.$$

for a,b∈R provided that this integral is well defined.

(ii) **(8 分)** Define the integral equation as follows.

$$F(x) = \int_{-1}^{5} f_1(x) f_2(y) + f_1(y-2) f_2(x) dy$$

where $f_1(x) = x^9 e^{x^2}$ and $f_2(x) = \frac{\sin^4(x-1)+\sin^4(x+5)}{\sin^4(x-1)+\sin^4(x-3)+\sin^4(x-9)+\sin^4(x+5)}$.

Calculate the value of $F(1)$.

3. Let $f:[a,b]\times[c,d]\to \mathrm{R}$ be a continuous function and $\frac{\partial f(x,y)}{\partial x}$ exists and is continuous on $[a,b]\times[c,d]$. Then one can get the famous rule

$$\frac{d}{dx}\left[\int_c^d f(x,y)dy\right]=\int_c^d \frac{\partial f(x,y)}{\partial x}dy \text{ (Leibnitz's rule)}$$

(i) (10 分) [General Leibnitz' s rule] Use these facts to prove that

$$\frac{d}{dx}\left[\int_{\varphi(x)}^{\psi(x)} f(x,y)dy\right]=\int_{\varphi(x)}^{\psi(x)}\frac{\partial f(x,y)}{\partial x}dy+f(x,\psi(x))\cdot\frac{d\psi}{dx}-f(x,\varphi(x))\cdot\frac{d\varphi}{dx}.$$

(ii) (8 分) Show that

$$\int_0^{x^2} 4\tan^{-1}\left(\frac{y}{x^2}\right)dy = x^2(\pi-2\ln 2).$$

4. Determine the convergence of the series and integral.

(i) (7 分) $\sum_{n=1}^{\infty}\left[\left(\frac{1+n}{1+n^2}\right)^2+\frac{2n(2n-1)}{(2n+1)^2(2n+2)^2}\right]$.

(ii) (5 分) $\int_1^{\infty}\frac{\sin^p x}{x^p}dx, p>1$.

5. (10 分) Suppose that Q be the solid region bounded by the coordinate planes, the paraboloid $z=2(x^2+y^2)$ and the cylinder $x^2+y^2=1$ with $x,y,z\geq 0$. Let $\mathbf{F}(x,y,z)=(xz,x^2y,y^2z)$ be defined on the surface S oriented by a unit normal vector $\mathbf{N}$ where S is the surface of Q. Evaluate the surface integral

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,dS.$$

6. Let V_1 and V_2 be two subspaces of the vector space V. Prove that

(i)(10 分) $V_1 \cap V_2$ is a subspace of V.

(ii)(10 分) $V_1 + V_2$ is a subspace of V.

7. Let V be the vector space of all polynomials $p(x)$ with degree at most two, and let $f: V \to V$ be the linear transformation $f(p(x)) = p(x) + xp'(x)$.

(i)(5 分) Prove that $1+x$, $1-x$, $1+x^2$ form a basis of V.

(ii) (10 分) Find the matrix representation $[f]_\beta$ of f with respect to the ordered basis $\beta = \{1+x, 1-x, 1+x^2\}$.

8. (20 分) Find the eigenvalues and the corresponding eigenspaces for the matrices $A = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix}$

9. For the following matrix

$$A = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

(i)(10 分) Find a matrix P such that $P^{-1}AP$ is a diagonal.

(ii)(10 分) Determine an orthogonal matrix P such that $P^{-1}AP$ is a diagonal.