注意：考試開始鈴響前，不得翻閱試題，
並不得書寫、畫記、作答。

國立清華大學 113 學年度碩士班考試入學試題

系所班組別：工程與系統科學系
乙組

科 目 代 碼 ：3001

考 試 科 目 ：工程數學

—作答注意事項—

1. 請核對答案卷（卡）上之准考證號、科目名稱是否正確。
2. 考試開始後，請於作答前先翻閱整份試題，是否有污損或試題印刷不清，得舉手請監試人員處理，但不得要求解釋題意。
3. 考生限在答案卷上標記「⬇由此開始作答」區內作答，且不可書寫姓名、准考證號或與作答無關之其他文字或符號。
4. 答案卷用盡不得要求加頁。
5. 答案卷可用任何書寫工具作答，惟為方便閱卷辨識，請儘量使用藍色或黑色書寫；答案卡限用 2B 鉛筆畫記；如畫記不清（含未依範例畫記）致光學閱讀機無法辨識答案者，其後果一律由考生自行負責。
6. 其他應考規則、違規處理及扣分方式，請自行詳閱准考證明上「國立清華大學試場規則及違規處理辦法」，無法因本試題封面作答注意事項中未列明而稱未知悉。

1. Find the solution of the given differential equation. Your answer should include the general solution $y(x)$ and the exact solution when an initial or boundary condition is provided.

(a) $x\frac{dy}{dx} - 2y = x^4 e^x.$ (5 %)

(b) $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -9x^2,\ y(1) = 7$ and $y(2) = 2\ln 2$ (6 %)

(c) $\frac{dy}{dx} = 2x + y - 2,\ y(0) = 2.$ (6 %)

2. Find the series solution of the following differential equation about x = 0.

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy = 0.$$ (10%)

You have to express the solution in the form of $y(x) = C_1y_1(x) + C_2y_2(x)$. To save time, you can only show the first four terms of $y_1(x)$ and $y_2(x)$.

3. Use the Laplace transform to solve the problem and obtain $y(t)$.

$$\frac{dy}{dt} + \int_0^t y(\tau)d\tau = u(t-1) - u(t-2),$$ (8 %)

where $y(0) = 0$ and $u(t)$ is the unit step function.

4. Consider the matrix

$$M = \begin{bmatrix} 5 & 6 & -3 \\ -2 & -3 & 2 \\ 2 & 2 & 0 \end{bmatrix}.$$

(a) Find the determinant of M and obtain the inverse matrix M^{-1} (7 %).

(b) Estimate the eigenvalues and eigenvectors of M (8 %).

5. Solve the wave equation $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$ subject to the conditions:

$$u(0,t) = 0, \quad u(\pi,t) = 0, \quad t > 0$$

$$u(x,0) = \frac{\pi}{2}x, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad 0 < x < \pi$$

(10%)

6. Use Fourier cosine transform to solve the steady-state temperature $u(x,y)$ in a plate from the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ x \geq 0, y \geq 0$ subject to the conditions:

$u_y(x,0) = 0, x \geq 0$ and $u(0,y) = \begin{cases} 30, & 0 \leq y \leq \pi \\ 0, & y > \pi \end{cases}$ (10%)

7. Solve temperature $u(r,t)$ in a circular plate of radius c which is determined from the heat equation (10%)

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)$$

$$u(c,t) = 0, \quad t > 0$$

$$u(r,0) = f(r), \quad 0 < r < c$$

[Hint: A general solution of the Bessel equation $x^2 y'' + xy' + (\alpha^2 x^2 - v^2)y = 0$ is $y = AJ_v(\alpha x) + BY_v(\alpha x)$.]

8. (a) Use Stokes' theorem to determine the line integral of the vector function $\mathbf{F} = (2x + y)\mathbf{i} + (x + y + z)\mathbf{j} + (y + 2z)\mathbf{k}$ along a closed curve C: $(2\cos\theta + \sin\theta)\mathbf{i} + (\cos\theta - 2\sin\theta)\mathbf{j} + 3\sin\theta\,\mathbf{k},\ 0 \leq \theta < 2\pi$ (5%)

(b) Find the Laurent series of the complex function $f(z) = \frac{1}{(z-i)^2(z+2i)}$, $|z - i| > 3$

(5%)

(c) Determine the integral $\oint_C \frac{\exp(z)}{z^3+z^2} dz$ where C is the circle of $|z| = 2$, oriented counterclockwise. (5%)

(d) Evaluate $\int_0^{2\pi} \frac{1}{(2+\cos\theta)^2} d\theta$ (5%)