注意：考試開始鈴響前，不得翻閱試題，
並不得書寫、畫記、作答。

國立清華大學 110 學年度碩士班考試入學試題

系所班組別：聯合招生

科目代碼：9801

考試科目：工程數學

一作答注意事項一

1. 請核對答案卷（卡）上之准考證號、科目名稱是否正確。
2. 考試開始後，請於作答前先翻閱整份試題，是否有污損或試題印刷不清，得舉手請監試人員處理，但不得要求解釋題意。
3. 考生限在答案卷上標記「由此開始作答」區內作答，且不可書寫姓名、准考證號或與作答無關之其他文字或符號。
4. 答案卷用盡不得要求加頁。
5. 答案卷可用任何書寫工具作答，惟為方便閱卷辨識，請儘量使用藍色或黑色書寫；答案卡限用 2B 鉛筆畫記；如畫記不清（含未依範例畫記）致光學閱讀機無法辨識答案者，其後果一律由考生自行負責。
6. 其他應考規則、違規處理及扣分方式，請自行詳閱准考證明上「**國立清華大學試場規則及違規處理辦法**」，無法因本試題封面作答注意事項中未列明而稱未知悉。

國立清華大學 110 學年度碩士班考試入學試題

系所班組別：聯合招生 (0598)

考試科目（代碼）：工程數學 (9801)

 *請在【答案卷】作答

1. Solve the differential equations.

(a) $\frac{dx}{dy} = \frac{-y-6xy^2}{2y^3+x^2}$. (5%)

(b) $\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) = 9x$. Obtain $y(x)$ that subjects to $y(1) = \frac{1}{2}$, $y'(1) = -3$. (5%)

2. Solve the system of differential equations for $x(t)$ and $y(t)$.

$$\frac{dx}{dt} - 2x - y = e^t$$

$$-2x + \frac{dy}{dt} - 3y = 3e^t$$ (10%)

3. Find the series solution of the following differential equation about x = 0.

$$2x\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + xy = 0 \ .$$ (10%)

You have to express the solution in the form of $y(x) = C_1y_1(x) + C_2y_2(x)$. To save time, you can only show the first five terms of $y_1(x)$ and $y_2(x)$.

4. Use the Laplace transform to solve the problem and obtain $y(t)$.

$$\frac{dy}{dt} = 1 - \sin t - \int_0^t y(\tau)\,d\tau \text{ and } y(0) = 0.$$ (5%)

5. Consider the matrix $M = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ -1 & 1 & 3 \end{bmatrix}$.

(a) Find the determinant of M and obtain the inverse matrix M^{-1}. (3%+4%)

(b) Estimate the eigenvalues and eigenvectors of M. (4%+4%)

6. (a) Let $f(x,y) = \alpha\sin(xy) + x^2 + 4y^2(1-y)$. For what values of α will f have a local minimum at $(0,0)$? (5%)

(b) Let $I(a) = \int_0^1 \frac{x^a - 1}{\ln x}dx$, $(a \geq 0)$. Find $I'(a)$. It is evident that $I(0) = 0$. Solve $I(a)$. (5%)

國立清華大學 110 學年度碩士班考試入學試題

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7. (a) Apply the divergence theorem to evaluate $\oiint_S (\mathbf{F}\cdot\mathbf{n})dS$ where $\mathbf{F} = 2r^3\hat{\boldsymbol{e}}_r - r\hat{\boldsymbol{e}}_\theta + 3\theta\hat{\boldsymbol{e}}_z$ and S is the surface of the region bounded by the cylinder: $r \le 5$, $0 \le \theta \le 2\pi$, $0 \le z \le 4$. (5%)

(b) Let $\mathbf{F} = 3r\hat{\boldsymbol{e}}_r - 2rz^2\hat{\boldsymbol{e}}_\theta + 4r^2\hat{\boldsymbol{e}}_z$. Find $\nabla\times\mathbf{F}$. Evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$ where C is a counterclockwise circle $x^2 + y^2 = 9$ on the xy plane. (5%)

[Formula] Divergence and curl in cylindrical coordinates:

$$\nabla\cdot\mathbf{v} = \frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{\partial v_z}{\partial z}$$

$$\nabla\times\mathbf{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial\theta} - \frac{\partial v_\theta}{\partial z}\right)\hat{\boldsymbol{e}}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{\boldsymbol{e}}_\theta + \frac{1}{r}\left(\frac{\partial(rv_\theta)}{\partial r} - \frac{\partial v_r}{\partial\theta}\right)\hat{\boldsymbol{e}}_z$$

8. (a) Let $f(x) = \begin{cases} A_0, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$, $A_0 > 0$.

The Fourier integral representation of $f(x)$ is $f(x) = \int_0^\infty [a(\omega)\cos\omega x + b(\omega)\sin\omega x]dx$. Find $a(\omega)$ and $b(\omega)$. (5%)

(b) Consider an infinite beam problem $EIu'''' + ku = f(x)$ with the loading $f(x)$ given in (a). The deflection u can be solved and written as $u(x) = \int_0^\infty [c(\omega)\cos\omega x + d(\omega)\sin\omega x]dx$. Find $c(\omega)$ and $d(\omega)$. (5%)

9. Solve the Dirichlet problem in polar coordinates: $\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}=0$, $1 \le r \le 2$, $0 \le \theta \le \frac{\pi}{4}$, with $u(r,0) = 0$, $u\left(r,\frac{\pi}{4}\right) = 0$, $u(1,\theta) = 0$, $u(2,\theta) = 10$.

(10%)

10. (a) Evaluate $\oint_C \frac{z-1}{(z+1)(z-2)^2}dz$ where C is the counterclockwise circle $|z| = 4$.

(5%)

(b) Evaluate the integral $\int_0^\infty \frac{\ln x}{x^2+4}dx$ by residue theorem. (5%)