注意：考試開始鈴響前，不得翻閱試題，
並不得書寫、畫記、作答。

國立清華大學 111 學年度碩士班考試入學試題

系所班組別：計量財務金融學系
乙組(財務工程組)

科 目 代 碼 ：5103

考 試 科 目 ：微積分

—作答注意事項—

1. 請核對答案卷（卡）上之准考證號、科目名稱是否正確。
2. 考試開始後，請於作答前先翻閱整份試題，是否有污損或試題印刷不清，得舉手請監試人員處理，但不得要求解釋題意。
3. 考生限在答案卷上標記「↓ 由此開始作答」區內作答，且不可書寫姓名、准考證號或與作答無關之其他文字或符號。
4. 答案卷用盡不得要求加頁。
5. 答案卷可用任何書寫工具作答，惟為方便閱卷辨識，請儘量使用藍色或黑色書寫；答案卡限用 2B 鉛筆畫記；如畫記不清（含未依範例畫記）致光學閱讀機無法辨識答案者，其後果一律由考生自行負責。
6. 其他應考規則、違規處理及扣分方式，請自行詳閱准考證明上「**國立清華大學試場規則及違規處理辦法**」，無法因本試題封面作答注意事項中未列明而稱未知悉。

國立清華大學 111 學年度碩士班考試入學試題

系所班組別：計量財務金融學系碩士班 乙組

考試科目（代碼）：微積分（5103）

共__1__頁，第__1__頁　　　*請在【答案卷、卡】作答

Problem 1 (20%).
(*i*)[5%] Let A_0 and r be nonnegative constants. For any $t > 0$, evaluate $\lim_{m\to\infty} A_0\left(1+\frac{r}{m}\right)^{mt}$.
(*ii*)[5%] Determine the convergence or divergence of the series $\sum_{n=1}^{\infty}\frac{1}{n}$. Justify your answer.
(*iii*)[5%] Determine the convergence or divergence of the series $\sum_{n=1}^{\infty}\frac{1}{n^{1.00000001}}$. Justify your answer.
(*iv*)[5%] Determine if $\lim_{(x,y)\to(0,0)}\frac{x^2-y^2}{x^2+y^2}$ exists or not. Justify your answer.

Problem 2 (10%). The n-th order Taylor approximation for a function f about $x = a$ is given by

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

where the right-hand side of the above approximation is called the *n-th order Taylor polynomial* for f about $x = a$. Now, let the function $f(x)$ be given by the conditions $x(0) = 1$ and

$$\frac{d}{dx}f(x) = xf(x) + 2(f(x))^2.$$

Determine the second-order Taylor polynomial for $f(x)$ about $x = 0$.

Problem 3 (10%).
(*i*)[5%] Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ with $f(x,y) := 4x^2 + 10y^2$ where $D := \{(x,y) : x^2+y^2 \le 4\}$. Find the maximum and minimum values of f over D.
(*ii*)[5%] Find the corresponding maximum and minimum points that achieve the maximum and minimum values, respectively.

Problem 4 (15%). For $0 < c \le 1$, consider the function $u_c : \mathbb{R}_+ \to \mathbb{R}$ with $u_c(x) := \frac{x^c-1}{c}$ where $\mathbb{R}_+ := \{x \in \mathbb{R} : x \ge 0\}$.
(*i*)[5%] Show that for $x > 0$, $\lim_{c\to 0} u_c(x) = \log x$.
(*ii*)[10%] Show that $u_c(x)$ is *strictly increasing* and *concave*.

Problem 5 (20%). Consider a function

$$H(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}dy.$$

(*i*) Show that $H(-\infty) = 0$.
(*ii*) Show that $H(\infty) = 1$.
(*iii*) Show that $H(0) = 1/2$.
(*iv*) Show that $H(-x) = 1 - H(x)$.

Problem 6 (10%). Suppose $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ with $f(\mathbf{x}) = f(x_1,\ldots,x_n)$, let g be a function of one variable defined over the range of f, and let $\mathbf{c} \in S$. Define $h(\mathbf{x}) := g(f(\mathbf{x}))$. Show that if g is increasing and $\mathbf{c}$ maximizes f over S, then $\mathbf{c}$ is also maximizes h over S.

Problem 7 (15%). Consider an integral of the form

$$\lim_{T\to\infty}\int_{t_0}^{T} U(c(t))e^{-\alpha t}dt$$

where $c(t)$ and $U(\cdot)$ are continuous functions and $\alpha > 0$. Suppose that there exist numbers M and β with $\beta < \alpha$ such that $|U(c(t))| \le Me^{\beta t}$ for all $t \ge t_0$ and for each possible $c(t)$ at time t. Show that the integral above converges.