1. (12 pts) Let $\{a_n\}$ be a sequence such that

$$a_1 + a_2 + \cdots + a_n = \frac{n^2}{2n+1}$$

for all n. Does $\lim_{n \to \infty} a_n$ exist? If so, find its value.

2. (12 pts) Among all the ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

that pass through the point $(3,4)$, which one has the smallest area?

3. (12 pts) Find a continuous function f such that

$$f(x) = 2 + \int_0^x \frac{f(t)}{(t+2)(t+3)} dt$$

for all $x \geq 0$.

4. (12 pts) Find the sum of the series

$$\sum_{n=0}^{\infty} (\ln 2)^n \left(1 + \frac{1}{n!}\right).$$

5. (12 pts) Solve the initial value problem:

$$xy' - y = x^2 \cos x, \qquad y(\pi/2) = 0.$$

6. (12 pts) Show that if $a > -1$ and $b > a + 1$, then the following integral is convergent

$$\int_0^{\infty} \frac{x^a}{1+x^b} dx.$$

7. (14 pts) Suppose that p is a polynomial of degree n so that

$$\int_k^{k+1} p(x)\, dx = 0$$

for $k = 0, 1, 2, \cdots, n-1$. Evaluate

$$\frac{\int_n^{n+1} p(x)\, dx}{\int_{n+1}^{n+2} p(x)\, dx}.$$

8. (14 pts) Compute the limit

$$\lim_{t \to 0} \left(\int_0^1 (1+x)^t\, dx\right)^{1/t}.$$

務必書寫計算過程，否則不予計分。