國 立 清 華 大 學 命 題 紙

97 學年度 統計學研究所 碩士班入學考試

科目 機 率 論 科目代碼 0102 共 2 頁第 1 頁 *請在【答案卷】內作答

請在答案卷內詳細寫出計算或導証過程

1. (15%) Consider k urns U_i, i=1, ..., k, each of which contain m white balls and n black balls. A ball is drawn at random from urn U_1 and is placed in urn U_2. Then a ball is drawn at random from urn U_2 and is placed in urn U_3 etc. Finally, a ball is chosen at random from urn U_{k-1} and is placed in urn U_k. A ball is then drawn at random from urn U_k. Compute the probability that this last ball is black.

2. (20%) Suppose that X and Y are two jointly distributed random variable with joint probability density function:

$$f(x,y) = \begin{cases} c \cdot xy(1-x), & \text{for } 0 < x < 1,\ 0 < y < 1, \\ 0, & \text{otherwise,} \end{cases}$$

where c is a constant.

(a) Find the value of c.

(b) Find the marginal probability density function of X and Y.

(c) Find the conditional probability density function of Y given X=x.

(d) Are X and Y independent? Explain your answer.

(e) Find the probability $P(Y < 1/2 | X > 1/2)$.

3. (10%) Consider certain events which in every time interval $[t_1, t_2]$ $(0 < t_1 < t_2)$ occur according to the Poisson distribution $P(\lambda(t_2 - t_1))$. Let T be the random variable denoting the time which lapses between two consecutive such events. Derive and identify the distribution of T by computing the probability that T>t.

4. (15%) Let X_i, $i = 1, \ldots, n$, be independent random variables such that X_i has continuous and strictly increasing cumulative distribution function F_i. Set $Y_i = F_i(X_i)$, $i = 1, \ldots, n$. Show that the random variable

$$Z = -2 \sum_{i=1}^{n} \log(1 - Y_i)$$

is distributed as χ^2_{2n}.

5. (15%) The number of people who enter an elevator on the ground floor is a Poisson random variable with mean 10. If there are N floors above the ground floor and if each person is equally likely to get off at any one of these N floors, independently of where the others get off, compute the expected number of stops that the elevator will make before discharging all of its passengers.

6. (10%) Let X be a random variable with moment generating function $M(t)$ and set

$$K(t) = \log(M(t))$$

for those t's for which $M(t)$ exists. Furthermore, suppose that $E(X) = \mu$ and $Var(X) = \sigma^2$ are both finite. Then show that

(a) $\left.\frac{d}{dt}K(t)\right|_{t=0} = \mu,$

(b) $\left.\frac{d^2}{dt^2}K(t)\right|_{t=0} = \sigma^2.$

7. (15%) Let X_i, $i = 1, \ldots, n$, be independent random variables distributed as Uniform(0, 1) and set

$$Y_n = \max(X_1, \ldots, X_n) \text{ and } Z_n = n(1 - Y_n).$$

Then show that, as $n \to \infty$, one has:

(a) Y_n converges in probability to one,

(b) Z_n converges in distribution to Z, where Z has the Exponential distribution with parameter $\lambda = 1$.