1. Let $X_1, \ldots, X_n$ be iid sample observations from a population with cdf $F(x)$, where $-\infty < x < \infty$. Define a random variable

$$W \equiv \text{the number of } X_1, \ldots, X_n \le c,$$

where c is a given constant.

(a) (3%) Find the distribution of W.

(b) (3%) Find $E(W)$ and $Var(Y)$.

2. Assume that the random vector $(X, Y)'$ follows a bivariate normal distribution with mean vector $(\mu_1, \mu_2)'$ and covariance matrix $\begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$, where

$-\infty < \mu_1, \mu_2 < \infty$, $\sigma_1^2 > 0$, $\sigma_2^2 > 0$, and $\rho > 0$.

(a) (3%) Find a necessary and sufficient condition under which $X + Y$ and $X - Y$ are independent.

(b) (5%) Derive the conditional distribution of X given $Y = y$.

3. Suppose $X_1, \ldots, X_n$ are iid sample observations from a normal distribution with mean μ and variance 4.

(a) (3%) Find the UMVUE of μ^2.

(b) (4%) Does the variance of the UMVUE you obtained in (3.a) reach the Cramer-Rao lower bound? Explain.

4. (7%) Given $\alpha = \beta = 0.10$, derive a sequential probability ratio test (SPRT) for testing $H_0 : \mu = 10$ against $H_1 : \mu = 12$ if one is sampling from a normal distribution with mean μ and variance 4.

5. Let X be a single observation from a Binomial distribution with parameters n and p. Consider the class of estimators $\mathfrak{D} = \{d(X) = cX \mid -\infty < c < \infty\}$.

   (a) (4%) If the loss function is a squared error loss function, i.e., $L(p, d) = (d-p)^2$. Find the risk functions of the estimators in $\mathfrak{D}$.

   (b) (7%) Find the minimax estimator of p in the class $\mathfrak{D}$.

   (c) (5%) As a prior distribution of p, we shall use a Beta(a, b), where a, $b > 0$. Under the squared error loss defined in (5.a), find the Bayes estimator of p.

6. Let X be a discrete variable whose probability mass functions under H_0 and H_1 are given in the following table.

$X = x$	1	2	3	4	5	6	7
$f(x\|H_0)$	0.01	0.02	0.03	0.05	0.05	0.07	0.77
$f(x\|H_1)$	0.03	0.09	0.10	0.10	0.20	0.18	0.30

   (a) (5%) List all the critical regions of size $\alpha = 0.10$.

   (b) (5%) Among the critical regions listed in (6.a), which one has the smallest type-II risk?

7. (a) (5%) Let

$$Y_1 = 2\theta - \phi + \varepsilon_1$$
$$Y_2 = \theta + \varepsilon_2$$
$$Y_3 = \theta + 2\phi + \varepsilon_3$$

where $E(\varepsilon_i) = 0$, for $i = 1, 2, 3$. Find the least squares estimators of θ and ϕ.

   (b) (5%) Let $Y_i = a + bx_i + \varepsilon_i$, $i = 1, 2, \ldots, n$, where $\varepsilon_1, \ldots, \varepsilon_n$ are mutually independent such that $E(\varepsilon_i) = 0$ and $Var(\varepsilon_i) = \sigma^2$. What is a necessary and sufficient condition that the least squares estimators of a and b are uncorrelated?

8. (7%) The following table contains random sample observations from a non-normal distribution taken at two different times, I and II. There is reason to believe that the median of the distribution may have shifted to the right between the sampling periods. Do these data justify that belief on the base of the rank-sum test?

I	72	21	32	52	73	40	31	62	40	42					
II	24	56	50	30	53	78	33	70	15	29	36	60	57	79	51

9. Suppose we have a linear model (subscripts refer to the dimensions of the corresponding matrices or vectors)

$$y_{T\times 1} = X_{T\times K}\beta_{K\times 1} + \varepsilon_{T\times 1},$$

with $E(\varepsilon) = 0$, $Cov(\varepsilon) = \sigma^2 I_T$, and rank$(X) = r$.

(a) (5%) Describe and prove a necessary condition under which β is estimable, i.e., an unbiased estimator exists.

(b) (5%) If $\Phi_{n\times 1} = H'_{n\times K}\beta_{K\times 1}$ contains n estimable functions and $\hat{\Phi}$ is the ordinary least squares (OLS) estimator of Φ. Find $Cov(\hat{\Phi})$.

10. Suppose that $X_1, \ldots, X_n$ are iid sample observations from a normal distribution whose mean μ and variance σ^2 are unknown. Let $p = \Pr(X_1 \le u)$.

(a) (5%) If p is given, find the UMVUE of u.

(b) (7%) If u is given, find the UMVUE of p.

(c) (7%) Find the UMVUE of the density function of X_1 at u.