1. Evaluate the following:

10% (a) $\dfrac{d}{dx}(x+1)^{\log_2 x}$.

10% (b) $\displaystyle\int_0^1 f(x)\,dx$, where $\displaystyle\int_0^1 \left(\dfrac{1}{\sqrt{9-4x^2}}+f(x)\right)dx = 2$.

2.

5% (a) Prove that

$$-2 < \frac{e^{-x^2}-1}{x} < 0 \qquad (0 < x < 1).$$

5% (b) Find the value of $\displaystyle\lim_{x\to 0}\frac{e^{-x^2}-1}{x}$.

3.

5% (a) Find the Taylor series expansion of $\tan^{-1} x$ for $|x| < 1$.

5% (b) Use (a) to prove

$$\pi = \frac{6}{\sqrt{3}}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{1}{3}\right)^n.$$

4.

5% (a) The integral

$$I = \int_0^1 \int_0^{\sqrt{1-y^2}} \sqrt[5]{1+x^2+y^2}\,dx\,dy$$

can be rewritten as a double integral of the form:

$$\iint_R f(x,y)\,dA.$$

Find $f(x,y)$ and sketch the region R.

5% (b) Find the value of I given in (a).

5. Set $\vec{a} = (1, 2, -3)$, $\vec{b} = (1, 0, 1)$, and $\vec{c} = (0, 1, 1)$.

8% (a) Find $\alpha, \beta \in \mathbf{R}$ such that $\vec{a} - \alpha\vec{b} - \beta\vec{c}$ is orthogonal to both of $\vec{b}$ and $\vec{c}$.

7% (b) Find the distance from the point $(1, 2, -3)$ to E, where E is the plane determined by the two vectors $\vec{b}$ and $\vec{c}$.

6. Let $T : \mathbf{R}^3 \mapsto \mathbf{R}^2$ be linear. Assume that

$$T(\vec{a}) = (0, 1), \quad T(\vec{b}) = (-1, 5), \quad T(\vec{c}) = (3, 2),$$

where $\vec{a} = (1, 1, 0)$, $\vec{b} = (0, 1, 1)$, and $\vec{c} = (1, 0, 1)$.

8% (a) Find the null space of T.

7% (b) Let V be the space spanned by $\vec{a}$ and $\vec{b}$. Define $\tilde{T} : V \mapsto \mathbf{R}^2$ by

$$\tilde{T}(x\vec{a} + y\vec{b}) = xT(\vec{a}) + yT(\vec{b}) \qquad (x, y \in \mathbf{R}).$$

Is $\tilde{T}$ invertible? Why?

7. Let

$$A = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}.$$

5% (a) Find A^{-1}.

5% (b) Find the eigenvalues and eigenvectors of A.

5% (c) Find a 2×2 matrix S such that SAS^{-1} is a diagonal matrix.

5% (d) Set

$$A^k = \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix} \qquad (k = 1, 2, \cdots).$$

Find $\lim_{k \to \infty} a_k$.