This examination contains FIVE questions on three pages. Please put down detail calculations neatly and BOX your final answer clearly on the answer sheet. Answers without detail calculations would be disregarded!

1. Quantum Numbers (10%)

Consider a free particle in two dimensions that is described by the Hamiltonian

$$H = \frac{1}{2m}\left(P_x^2 + P_y^2\right). \tag{1}$$

Show that the Hamiltonian commute with P_x, P_y and L_z. Can we use all three quantum numbers (p_x, p_y, l_z) to label the eigenstates?

2. Spin Precession (20%)

Apply a magnetic field along the $x-$axis to a free $s = 1/2$ spin. The Hamiltonian can be written down as a 2×2 matrix

$$H = g\mu_B \vec{H} \cdot \vec{S} = \frac{\hbar\omega}{2}\sigma_x, \tag{2}$$

where $\omega = g\mu_B H$ and σ_x is one of the Pauli matrices.

(a) (10%) Write down equations of motion in Heisenberg picture and solve for the time-dependent operators $\vec{S}(t)$.

(b) (10%) If the initial state is $|s_z = 1/2\rangle$, evaluate the average spin $\langle \vec{S(t)} \rangle$ as a function of time.

3. Particle in Magnetic Field (25%)

In the presence of a magnetic field, the eigenstates of a free particle in two dimensions are referred as Landau levels. Suppose the particle is in one of these states and the wave function is given as

$$\psi(x, y) = N(x + iy)e^{-\frac{x^2+y^2}{2}}. \tag{3}$$

(a) (5%) Determine the normalization constant N.

(b) (10%) Compute the current density $j(x)$.

(c) (5%) Since $j(x) = \rho(x)v(x)$, where $\rho(x)$ is the particle density, the velocity field can be determined from current density. Furthermore, make use of the identity $L_z = m\vec{r}\times\vec{v}$ to evaluate the angular momentum L_z.

(d) (5%) Now compute the angular momentum through a different approach. Make use of $\vec{L} \equiv \vec{R} \times \vec{P}$ in polar coordinate to evaluate the angular momentum again and compare with previous result.

4. One-Dimensional Quantum Wire (20%)

A fancy to confine electrons to one dimension is simply by a parabolic potential in the transverse direction. The confined one-dimensional electron gas is called quantum wire and can be modeled by the Hamiltonian

$$H = \frac{P_x^2}{2m} + \frac{P_y^2}{2m} + \frac{1}{2}kY^2 - \frac{\hbar^2}{2ma}\delta(X), \qquad (4)$$

where $a > 0$ representing an attractive impurity at the center of the wire.

(a) (5%) Estimate the rough width of the quantum wire.

(b) (5%) Evaluate the bound-state energy due to the attractive impurity.

(c) (10%) Compute the linear density along the wire $n(x) = \int dy|\psi(x,y)|^2$.

5. Identical Particles (25%)

Place two $s = 1/2$ fermions inside an infinitely-deep well between $x = 0$ and $x = L$.

(a) (5%) Write down the ground state wave function $\Psi_0(x_1, s_1; x_2, s_2)$.

(b) (10%) Now apply a magnetic field that gives rise to the Zeeman interaction

$$H_b = -g\mu_B B(S_{1z} + S_{2z}). \qquad (5)$$

For sufficiently large magnetic field, the ground state is no longer the same as the previous wave function $\Psi_0(x_1, s_1; x_2, s_2)$. Find the new ground state wave function $\Psi_b(x_1, s_1; x_2, s_2)$ and locate the critical magnetic field where the ground state changes.

(c) (10%) Suppose now we add in the interaction between these two fermions

$$H_i = -\alpha\delta(x_1 - x_2). \tag{6}$$

Compute the ground state energy shift ΔE for both Ψ_0 and Ψ_b to the first order in perturbation theory. You might find the integral formula useful during calculation,

$$\int_0^1 dx \sin^{2n}(\pi x) = \frac{(2n)!}{4^n (n!)^2}.$$