注意：考試開始鈴響前，不得翻閱試題，並不得書寫、畫記、作答。

國立清華大學 108 學年度碩士班考試入學試題

系所班組別：數學系

考試科目(代碼)：高等微積分(0101)

一作答注意事項一

(1) (15%) If $a_1 = \sqrt{2}$, and

$$a_{n+1} = \sqrt{2 + \sqrt{a_n}}, \quad n = 1, 2, 3, \cdots.$$

prove that the sequence $\{a_n\}$ converges, and that $a_n < 2$ for $n = 1, 2, 3, \cdots$.

(2) Define a function $f : R^2 \to R$ by

$$f(x, y) = \begin{cases} 0, & (x, y) = (0, 0) \\ x^2 + y^2 - 2x^2 y - \frac{4x^6 y^2}{(x^4 + y^2)^2}, & (x, y) \neq (0, 0) \end{cases}$$

  (a) (5%) Prove that f is a continuous function on R^2.

  (b) (15%) Prove that the restriction of f to each line through $(0, 0)$ has a local minimum at $(0, 0)$.

  (c) (10%) Is $(0, 0)$ a local minimum for f? (Please give your reasons.)

(3) (10%) Prove that the function $f(x) = \ln x$ is not uniformly continuous on $(0, 1)$.

(4) (10%) For $n = 1, 2, 3, \cdots, x$ real, put

$$f_n(x) = \frac{x}{1 + nx^2}.$$

Show that $\{f_n\}$ converges uniformly to a function f. Does $\{f_n'\}$ converges to f'?

(5) (10%) Let D be the solid bounded by the cylinder $x^2 + y^2 = 4$, the plane $x + z = 6$, and the xy-plane. Find

$$\int\int_S F \cdot \vec{n} \, dS$$

where S is the boundary of D with the unit normal vector $\vec{n}$ directed outward from D and $F(x, y, z) = (x^2 + \sin z)i + (xy + \cos z)j + e^y k$.

(6) (a) (15%) Let $f : [0, 1) \to R$ be a one-to-one continuous function and A the range of f. Show that the inverse function $f^{-1} : A \to [0, 1)$ is continuous.

  (b) (10%) Please give a function $g : [0, 1) \to R^2$ which is one-to-one continuous. But its inverse, which is defined on the range of g, is not continuous.