注意：考試開始鈴響前，不得翻閱試題，並不得書寫、畫記、作答。

國立清華大學 108 學年度碩士班考試入學試題

系所班組別：計算與建模科學研究所

考試科目(代碼)：微積分(0301)

—作答注意事項—

1. 請核對答案卷（卡）上之准考證號、科目名稱是否正確。
2. 作答中如有發現試題印刷不清，得舉手請監試人員處理，但不得要求解釋題意。
3. 考生限在答案卷上標記「由此開始作答」區內作答，且不可書寫姓名、准考證號或與作答無關之其他文字或符號。
4. 答案卷用盡不得要求加頁。
5. 答案卷可用任何書寫工具作答，惟為方便閱卷辨識，請儘量使用藍色或黑色書寫；答案卡限用 2B 鉛筆畫記；如畫記不清（含未依範例畫記）致光學閱讀機無法辨識答案者，其後果一律由考生自行負責。
6. 其他應考規則、違規處理及扣分方式，請自行詳閱准考證明上「**國立清華大學試場規則及違規處理辦法**」，無法因本試題封面作答注意事項中未列明而稱未知悉。

國立清華大學 108 學年度碩士班考試入學試題

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考試科目（代碼）：微積分 (0301)

　*請在【答案卷、卡】作答

1. **(i)** (8 points) If the real numbers $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ satisfy $\sum_{i=1}^{n} x_i^2 = 1$ and, $\sum_{i=1}^{n} y_i^2 = 1$ please find the maximal value of $\sum_{i=1}^{n} x_i y_i$.

(ii) (7 points) Use (i) to show that the Cauchy's inequality,

$$\sum_{i=1}^{n} a_i b_i \leq \sqrt{\sum_{i=1}^{n} a_i^2} \cdot \sqrt{\sum_{i=1}^{n} b_i^2},$$

for the real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$.

2. Consider a function

$$f(a,b,c) = \frac{13a + 13b + 2c}{2a + 2b} + \frac{24a - b + 13c}{2b + 2c} + \frac{-a + 24b + 13c}{2c + 2a}$$

in $D = \{(a,b,c) \in \mathbb{R}^3 : a, b, c > 0\}$.

(i) (10 points) Show that $f(a,b,c)$ attains its minimum value in D. More precisely, prove that for $(a,b,c) \in D$, there holds

$$f(a,b,c) \geq \sqrt{20} + 19$$

and the equality holds only for $a = b = \frac{c}{2\sqrt{5}-1}$.

(ii) (5 points) Do there exist positive numbers b_0 and c_0 arriving at

$$f(1234^{5566}, b_0, c_0) = 2019\ ?$$

3. (10 points) Estimate the integral $\int_0^{0.5} e^{-x^2} dx$ accurate to within 0.001. Justify your answer.

4. Consider a sequence $\{x_n\}_{n \in \mathbb{N}}$ satisfying $x_1 = 1$ and $x_{n+1} = \left(\sum_{k=1}^{n} x_k\right)^{1/2}$.

(i) (5 points) Does $\sum_{k=1}^{n} \frac{(-1)^k}{x_k}$ converge as $n \to \infty$? Prove or disprove your assertion.

(ii) (10 points) Find all real number α such that the limit $\lim_{n \to \infty} \sum_{k=1}^{n} \frac{x_k}{k^\alpha}$ exists.

5. (i) (5 points) Is there a differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$f(0)f(2019) < 0 \text{ and } f(x)f'(x) \leq 0 \text{ for all } x \in (0,2019)?$$

If your answer is "YES", please give a "CLEAR" example. If "NO", please rigorously prove that there does NOT exist such a function f satisfying the above conditions.

(ii) (10 points) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a twice differentiable function and satisfy

$$f(0)f'(0) > 0 \text{ and } f(x)f''(x) \geq \left(f'(x)\right)^2 \text{ for all } x \in \mathbb{R}.$$

Prove that $f(5566)f(x)$ must be monotonically increasing and convex in $(0, \infty)$.

6. Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is a differentiable function. Please carefully think about the following statements.

(i) (5 points) If $a \in \mathbb{R}$ is an inflection point (反曲點) of f, then for any two distinct points $b, c \in \mathbb{R}$ there always holds

$$\frac{f(b) - f(c)}{b - c} \neq f'(a).$$

(ii) (5 points) If $a \in \mathbb{R}$ is not an inflection point of f, there must exist $b, c \in \mathbb{R}$ such that

$$\frac{f(b) - f(c)}{b - c} = f'(a).$$

(iii) (10 points) If f is twice differentiable at $a \in \mathbb{R}$ and $f''(a) \neq 0$, there must exist $b, c \in \mathbb{R}$ such that

$$\frac{f(b) - f(c)}{b - c} = f'(a).$$

If you think the statement is true, please prove it rigorously. If you think it is wrong, please give a counterexample.

Note. Don't confuse (i)-(iii) with the Mean Value Theorem.

7. (10 points) Prove that

$$\frac{\sin x \sin y}{xy} > \min\{\cos x, \cos y\} \text{ for any } x, y \in \left(0, \frac{\pi}{2}\right).$$

8. **(i)** (7 points) Prove that $\int_1^\infty \frac{sinx}{x}\,dx$ converges.

(ii) (8 points) Prove that $\int_1^\infty \left|\frac{sinx}{x}\right| dx$ diverges.

9. Evaluate the integrals

(i) (5 points) $\left(\int_0^1 \frac{dx}{\sqrt{1-x^4}}\right) \div \left(\int_0^1 \frac{dx}{\sqrt{1+x^4}}\right)$.

(ii) (5 points) $\iint_D (x-y)^2 dxdy$, where

$$D = \{(x,y) \in \mathbb{R}^2 : 4(x-y)^2 + (x+y)^2 \le 4\}.$$

10. **(i)** (8 points) Show that if f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$ and $D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \mathbf{u}$.

(ii) (7 points) Show that a differentiable function f decreases most rapidly at (x,y) in the direction opposite to the gradient vector, that is, $-\nabla f(x,y)$.

11. (10 points) Use the Divergence Theorem to evaluate the surface integral

$$\iint_S (2x + 2y + z^2)\,dS,$$

where $S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$.