國立清華大學 106 學年度碩士班考試入學試題

系所班組別：計算與建模科學研究所

考試科目（代碼）：微積分(0401)

　　*請在【答案卷】作答

(Note: （總分 150 分）Do not change rational or constant numbers (like 1/3 or π) to decimal numbers (0.333… or 3.1415…).)

1. (10 pts.) Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest away from the point (1,0).
2. (10 pts.) Find the area of the surface obtained by rotating the curve $x = \frac{1}{3}(y^2 + 2)^{3/2}$, $1 \le y \le 2$, about the x-axis.
3. (10 pts.) Show that the volume of a sphere of radius r is $V = 4\pi r^3/3$.
4. (10 pts.) Show by trigonometric substitution that the area of a circle with radius r is πr^2.
5. (10 pts.) Find the interval of convergence of the series:
   (a) $\sum_{n=1}^{\infty}\left(\frac{x^n}{\sqrt{n}}\right)$, (b) $\sum_{n=1}^{\infty} n!\,(2x-1)^n$.
6. (10 pts.) Find the minimum value of the function $f(x, y) = x^2 + y^2$ subject to the constraint $xy = 1$.
7. (10 pts.) State the Fundamental Theorem of Calculus and briefly prove it. (Hints: $F(x) = \int_a^x f(t)\mathrm{d}t$, $F'(x) = ?$, $\int_a^b f(x)\mathrm{d}x = ?$)
8. (10 pts.) Use a triple integral to find the volume of the tetrahedron enclosed by the coordinate planes (xy-plane, yz-plane, xz-plane) and the plane $2x + y + z = 4$.
9. (15 pts.) Let $f(x) = a^x$, where a is a positive number. (a) Show that $f'(x) = f'(0)a^x$, where $f'(0) = \ln a$. (b) Show that the number $e = 2.718281 \ldots$ can be obtained by taking the limit $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$. (Hints: $y = a^x$, $\frac{dy}{dx}$.)
10. (15 pts.) Evaluate the integrals (a) $\int \sin^3 x \cos^2 x\, \mathrm{d}x$ (b) $\int \ln x\, \mathrm{d}x$
    (c) $\int_1^2 \frac{4x^2-7x-12}{x^3-x^2-6x}\mathrm{d}x$.

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11. (15 pts.) Let $z = f(x, y)$ represent a surface S in the 3D space and $P_0(x_0, y_0, z_0)$ be a point on S. Let C_1 and C_2 be two curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$, respectively, with S through P_0. Derive the equation of the tangent plane to S at P_0 by the following method. Define $F(x, y, z) = f(x, y) - z = 0$ and $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$. Step 1. (7 pts.) Show that $\nabla F \cdot \mathbf{r}'(t) = 0$, where $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$. Step 2. (8 pts.) Derive the equation $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ using the gradient vector $\nabla F(x_0, y_0, z_0)$, where $f_x(x^0, y^0) = \left.\frac{\partial f(x,y)}{\partial x}\right|_{(x,y)=(x_0,y_0)}$.

12. (25 pts.) Consider the curve of the circular helix $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$ from the point (1, 0, 0) to the point (1, 0, 2π). (a) (5 pts.) Show that $ds = |\mathbf{r}'(t)|dt$, where s is the arc length parameter. (b) (5 pts.) Find the length of the curve. (c) (10 pts.) Find the unit tangent vector $\mathbf{T}\left(\frac{\pi}{2}\right)$ at $t = \frac{\pi}{2}$, the unit normal vector $\mathbf{N}\left(\frac{\pi}{2}\right)$, and the unit binormal vector $\mathbf{B}\left(\frac{\pi}{2}\right)$. (e) (5 pts.) Find the curvature κ of the curve at the point (1, 0, 0). (Hint: $\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$)