

1. Solve the following ordinary differential equation. Please give all steps in details, and give your answer in the explicit form.

(a) $y' + 2xy = 2x/y$, $y(0) = 2$. (8 points)

(b) $y'' + 3y' + 2y = 2u(t-2)$, $y(0) = 0$, $y'(0) = 0$. (8 points)

(c) $y'' + 4y = r(t)$, $y(0) = 0$, $y'(0) = 3$. (10 points)

$$r(t) = \begin{cases} 3\sin t & 0 < t < \pi \\ -3\sin t & t > \pi \end{cases}$$

2. A radioactive substance X, whose half-life is 1hr, decay to a stable substance Y. Please find the amount of X and Y as a function of time, assume the initial amount of X is A_0 and the initial amount of Y is 0. (14 points)

3. Matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

(a) Diagonalize A. Show the details of your work. (6 points)

(b) Use the result in (a) to find A^4 . (6 points)

Consider the following linear system

$$-2x_1 + 2x_2 - 3x_3 = 20$$

$$2x_1 + x_2 - 6x_3 = 4, \quad \text{which can be written as } \mathbf{Ax} = \mathbf{b}$$

$$-x_1 - 2x_2 = 1$$

(a) Determine the rank of the coefficient matrix A. (2 points)

(b) Find the inverse of the coefficient matrix A. (4 points)

(c) Find the solution. (4 points)

5. Evaluate $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} dA$, $\mathbf{F} = y\hat{\mathbf{i}} + x\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ $S: x^2 + y^2 = 1, -1 \leq z \leq 1$. (10 points)

6. Find the eigenvalues and the corresponding orthonormal set of eigenfunctions of the following problem. (10 points)

$$y'' + \lambda y = 0, \quad \text{B.C. } y(0) = 0, y(L) = 0.$$

7. Find the Fourier integral of the given function. (10 points)

$$f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & x < 0 \end{cases}$$

Use the result to evaluate $\int_0^{\infty} \frac{\cos wx + x \sin wx}{1+x^2} dx$.

8. Evaluate the following integral by using the residue theorem.

$$\oint_C \frac{4z-6}{z^3-4z^2+3z} dz, \text{ and } C: \left| z - \frac{1}{2} \right| = 1 \text{ clockwise. (8 points)}$$