國 立 清 華 大 學 命 題 紙

九十一學年度 原子科學 系（所） 甲 組碩士班研究生招生考試

科目 應用數學 科號 3202 共 2 頁第 1 頁 *請在試卷【答案卷】內作答

1. Solve the following initial value problems.

   (a) $[(x+1)e^x - e^y]dx = x\,e^y dy$, $y(1) = 0$ (6%)

   (b) $x^2y'' - 4xy' + 4y = 40x^{-4}$, $y(1) = 5, y'(1) = 0$ (6%)

   (c) $y'' + 9y = r(t)$, $y(0) = 1, y'(0) = 4$ (10%)

$$r(t) = \begin{cases} 8\sin t & 0 < t < \dfrac{\pi}{2} \\ 0 & t > \dfrac{\pi}{2} \end{cases}$$

2. Find the general solution of the following first order linear system $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}$,

   where $\mathbf{A} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$, and $\mathbf{g} = \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$. (10%)

3. For a 3×3 matrixt $\mathbf{A} = \begin{bmatrix} 4 & -6 & -6 \\ 0 & -2 & 0 \\ 1 & -1 & -1 \end{bmatrix}$.

   (a) Find a basis of eigenvectors of **A** that form an orthonormal system. (6%)

   (b) Diagonalize **A**. Show the details of your calculation. (6%)

4. Find the work done by **F** in the displacement along the curve C.

   $\mathbf{F} = yz\,\hat{\mathbf{i}} + zx\,\hat{\mathbf{j}} + xy\,\hat{\mathbf{k}}$, C: the intersection of $x^2+y^2+z^2 = 25$ and $z = y^2$. (8%)

5. Evaluate the surface integral $\iint_S \mathbf{F} \bullet \hat{\mathbf{n}} dA$, where $\hat{\mathbf{n}}$ is the outer unit normal vector.

   $\mathbf{F} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$, S: the surface of the cylinder $x^2+y^2 \le 1$, $0 \le z \le 1$., (10%)

6. The vibration of an elastic string is governed by the one-dimensional wave equation. Find the solution of the wave equation corresponding to the following conditions.

$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$, B.C. $u(0,t)=0$, $u(L,t)=0$ for all t; I.C. $u(x,0)=f(x)$, $\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$.

Where $f(x) = \begin{cases} \frac{2kx}{L} & 0 < x < \frac{L}{2} \\ \frac{2k}{L}(L-x) & \frac{L}{2} < x < L \end{cases}$. （20%）

7. Develop the given function in a Taylor series and find the radius of convergence.

$\frac{2+z}{1-z^2}$, center 0 （6%）

8. Evaluate the following integrals:

(a) $\oint_c \frac{2z+3}{z^2-z} dz$, C: $|z| = 2$ clockwise. （6%）

(b) $\int_0^{2\pi} \frac{d\theta}{13-5\sin\theta}$ （6%）