國立清華大學命題紙

99學年度工程與系統科學系乙組、核子工程與科學研究所甲組、聯合招生(工科丙組、先進光源工科組)碩士班入學考試

*請在【答案卷卡】作答

科目 工程數學 科目代碼 2701、2801、9801 共 2 頁，第 1 頁

1. Solve the ordinary differential equation $x^2y'' + 4xy' + (x^2+2)y = 0$, using Frobenius method. (10%)

2. Let's denote $J_\nu(x)$ and $Y_\nu(x)$ to be the Bessel functions of 1st kind and of 2nd kind, respectively. I recall you that $x^2y'' + xy' + (x^2 - \nu^2)y = 0$ is called Bessel's equation where ν is a real and nonnegative number. Find a general solution for the ordinary differential equation $xy'' + 11y' + xy = 0$ in terms of $J_\nu(x)$ and $Y_\nu(x)$.
[Hint: use the substitution $y = x^{-5}u$ in your derivation.]
(5%)

3. The Legendre polynomials of degree smaller than 6 are given as follow.

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x),$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \quad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

Develop $(x+1)^2$ in Fourier-Legendre series.
(5%)

4. Fourier transform of $f(x)$ is defined as $\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$. Find the Fourier transform of $f(x) = e^{-ax^2}$ $(a > 0)$.
(5%)

5. (a) Calculate the flux of vector
$\mathbf{F} = (x-y)\,\mathbf{i} + (y-z)\,\mathbf{j} + (z-x)\,\mathbf{k}$
out of the unit sphere.
(12%)

(b) Find a unit vector normal to the surface S given by
$z = x^2y^2 + y + 2$
at the point (1,0,1).
(13%)

6. The temperature in a rod of unit length in which there is heat transfer from both ends, $x=0$ and $x=1$, into a surrounding medium kept at a constant temperature 0 is determined from

$$k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t},\quad 0<x<1,\quad t>0,\quad k>0$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0}=h\,u(0,t),\quad h>0,\quad t>0$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=1}=-h\,u(1,t),\quad h>0,\quad t>0$$

The initial temperature in the rod is $f(x)$ throughout. Solve for $u(x,t)$.
(13%)

7. (a) Show that the value of an analytic function at any point z_0 is the average of its values on any circle, with z_0 as its center, which lies inside the region of analyticity.
(6%)

(b) Find all possible values of $(1+i)^i$.
(6%)

8. Laplace transform of $f(t)$ is defined as $L\{f(t)\}=\int_0^\infty e^{-st}f(t)dt$. Find the solution of the following equation using Laplace Transform.

$$y''+2y'+2y(t)=\delta(t-2),\quad y(0)=0,\ y'(0)=1$$

where $\delta(t-2)$ *is the unit impluse function*

$$\delta(t-2)=\begin{cases}\infty & \text{if } t=2\\ 0 & \text{otherwise}\end{cases}\qquad \text{and}\ \int_0^\infty \delta(t-2)dt=1,$$

and $L(\delta(t-a))=e^{-as}$.
(12%)

9. Obtain the general solution x(t), y(t) for the following coupled differential equations

$$x'=x(t)-3y(t).$$

$$y'=-x(t)-y(t)-4t^2$$

(13%)