國立清華大學命題紙

97學年度工程與系統科學系乙、丙組、先進光源科技學位學程乙組及核子工程與科學研究所甲組碩士班入學考試

科目工程數學 科目代碼 2901、3001、3201、3101 共 2 頁第 1 頁 *請在【答案卷卡】內作答

1. (a) Evaluate

$$\int_{(0,0)}^{(2,1)} (5y^3 + 20x^4 y^2)dx + (15xy^2 + 8x^5 y - 3)dy$$

along the path $x^4 - 6xy^3 = 4y^2$.

(5%)

(b) Let $\vec{v} = rz^2\hat{e}_z$, which is given in cylindrical coordinates (r, θ, z). Evaluate $\oint_S \hat{n}\cdot\vec{v}dA$ where S is the surface of the cone V (see figure).

(5%)

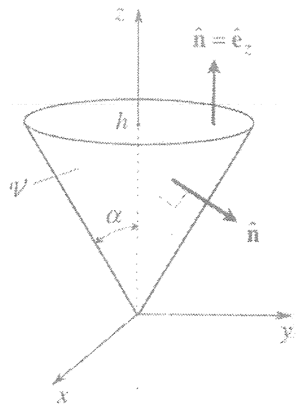

Figure The cone V

2. (a) Find an orthonormal set of the linear independent set {(2,0,0), (1,1,0), (3,3,3)} using Gram-Schmidt orthogonalization process. (5%)

(b) Matrix $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 4 & 3 \\ -1 & 3 & 4 \end{pmatrix}$ can be expressed as $Q^t DQ$ where D is a diagonal matrix and Q is an orthogonal matrix. Find D and Q. (5%)

3. Find the *harmonic* function $u(x, y)$ in the semi-infinite strip $0 < x < \pi$, $y > 0$ such that

$$u(0, y) = u(\pi, y) = 0 \qquad (y > 0),$$

$$u(x, 0) = 1 \qquad (0 < x < \pi),$$

and $|u(x, y)| < M$, where M is some constant.

(10%)

4. Determine the residue of each of the following functions at each singularity:

(a) $\tan z$,

(b) $\dfrac{\sin z - z}{z^6}$,

(c) $z\,e^{1/z}$.

(10%)

5. Solve by Fourier transform

$$u'''' + ku = w(x),$$

where k is constant and w(x) can be expanded in a Fourier integral, and $u(x), u'(x), u''(x), u'''(x) \to 0$, as $x \to \pm\infty$.

Hint: the Fourier transform of $f(x) = e^{-a|x|/\sqrt{2}} \sin\left(\frac{a}{\sqrt{2}}|x| + \frac{\pi}{4}\right)$ $(a > 0)$ is $\frac{2a^3}{\omega^4 + a^4}$.

(10%)

6. Find the weighting function of the following equation to become a SLP (Sturm-Liouville Problem) -type equation, $(1-x^2)y'' - xy' + \lambda y = 0$.

(5%)

7. Find the general solution $y(x)$ of the following differential equation

$$x^2 y'' - 2xy' + 2y = x \ln x.$$

(Hint: let $x = e^t$.)

(15%)

8. (a) Prove the following relations between Laplace transforms

$$L\{y'(t)\} = sL\{y(t)\} - y(0),$$

$$L\{t\, y(t)\} = -\frac{d}{ds} L\{y(t)\}.$$

(b) Solve the following problem using Laplace transform

$$t\, y'' + 2t\, y' + 2y = 0; \qquad y(0) = 0.$$

(15%)

9. Prove the recurrence relations satisfied by Legendre polynomials

$$(k+1)P_{k+1}(x) - (2k+1)\, x\, P_k(x) + k\, P_{k-1}(x) = 0. \qquad k = 1, 2, 3, \ldots$$

(Hint : A generating function of the Legendre polynomials is

$$(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n.$$

Differentiate the above equation once with respect to t.)

(15%)