

1. (a) Prove that the integral

$$\int_{-\infty}^{\infty} e^{-\lambda x^2} dx = \sqrt{\frac{\pi}{\lambda}}, \quad \lambda > 0$$

(7%)

(b) Evaluate the integral

$$\int_0^{\infty} x^n e^{-x^2} dx, \quad n \text{ is a positive integer.}$$

(8%)

2. Find the solution  $y(t)$  of the following differential equation:

$$y'' - y' - 12y = t \quad y(0) = -1, y'(0) = 0$$

(a) using Laplace transform,

(b) by combining the homogeneous and particular solutions.

(15%)

3. Find the general solutions  $y(x)$  and show the details:

(a)  $x^2 y'' + 2x y' - 6y = 0.$

(b)  $(1-x^2)y'' - 2xy' + 6y = 0, \quad -1 < x < 1.$

(10%)

4. Solve

$$u_{xx}(x,y) + u_{yy}(x,y) = 0, \quad \text{for } 0 < x < 1, 0 < y < 1,$$

$$u(x,0) = (1-x)^2,$$

$$u(x,1) = 0,$$

$$u_x(0,y) = 0,$$

$$u(1,y) = 0.$$

(15%)

5. Evaluate  $\int_S F \cdot dS,$

where vector  $F(x,y) = xy^2 \mathbf{i} + x^2y \mathbf{j} + y \mathbf{k}$  and  $S$  is the surface of the cylinder  $x^2 + y^2 = 2, -2 < z < 2,$  and  $x^2 + y^2 \leq 2$  when:  $z = \pm 2.$

(15%)

6. If  $A$  is a  $3 \times 3$  non-singular matrix with distinct eigenvalues  $\lambda_1$ ,  $\lambda_2$  and  $\lambda_3$ , and the corresponding column eigenvectors  $x_1$ ,  $x_2$  and  $x_3$ . Express the result of following problems in terms of  $\lambda_1$ ,  $\lambda_2$ ,  $\lambda_3$ ,  $x_1$ ,  $x_2$  and  $x_3$ .

(a)  $|A^n|$  (determinant of matrix  $A^n$ ),  $n$  is an integer. (7%)

(b)  $e^{At}$ ,  $t$  is a parameter. (8%)

7. Consider a complex function  $f(z)$  expanded as a Laurent series

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n \quad (1)$$

(a) find the region of convergence of series (1). (5%)

(b) Find the residue of  $f(z)$  at  $z=0$ . (5%)

8. A medical procedure has been shown to be effective in the early detection of an illness. A medical screening of the population is performed. The probability that the test correctly identifies someone with the illness as positive is 0.99, and the probability that the test correctly identifies someone without the illness as negative is 0.95. The incidence of the illness in the general population is 0.0005. You take the test, and the result is positive. What is the probability that you have the illness?

(5%)