

注意：考試開始鈴響前，不得翻閱試題，  
並不得書寫、畫記、作答。


國立清華大學 110 學年度碩士班考試入學試題

系所班組別：工程與系統科學系  
乙組

科目代碼：3101

考試科目：工程數學

### — 作答注意事項 —

1. 請核對答案卷（卡）上之准考證號、科目名稱是否正確。
2. 考試開始後，請於作答前先翻閱整份試題，是否有污損或試題印刷不清，得舉手請監試人員處理，但不得要求解釋題意。
3. 考生限在答案卷上標記  由此開始作答」區內作答，且不可書寫姓名、准考證號或與作答無關之其他文字或符號。
4. 答案卷用盡不得要求加頁。
5. 答案卷可用任何書寫工具作答，惟為方便閱卷辨識，請儘量使用藍色或黑色書寫；答案卡限用 2B 鉛筆畫記；如畫記不清（含未依範例畫記）致光學閱讀機無法辨識答案者，其後果一律由考生自行負責。
6. 其他應考規則、違規處理及扣分方式，請自行詳閱准考證明上「國立清華大學試場規則及違規處理辦法」，無法因本試題封面作答注意事項中未列明而稱未知悉。

國立清華大學 110 學年度碩士班考試入學試題

系所班組別：工程與系統科學系碩士班 乙組(0531)

考試科目（代碼）：工程數學 (3101)

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\*請在【答案卷】作答

1. Solve the differential equations.

(a)  $\frac{dx}{dy} = \frac{-y-6xy^2}{2y^3+x^2}$ . (5%)

(b)  $\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) = 9x$ . Obtain  $y(x)$  that subjects to  $y(1) = \frac{1}{2}$ ,  $y'(1) = -3$ . (5%)

2. Solve the system of differential equations for  $x(t)$  and  $y(t)$ .

$$\begin{aligned} \frac{dx}{dt} - 2x - y &= e^t \\ -2x + \frac{dy}{dt} - 3y &= 3e^t \end{aligned} \quad (10\%)$$

3. Find the series solution of the following differential equation about  $x = 0$ .

$$2x \frac{d^2y}{dx^2} + 3 \frac{dy}{dx} + xy = 0. \quad (10\%)$$

You have to express the solution in the form of  $y(x) = C_1y_1(x) + C_2y_2(x)$ . To save time, you can only show the first five terms of  $y_1(x)$  and  $y_2(x)$ .

4. Use the Laplace transform to solve the problem and obtain  $y(t)$ .

$$\frac{dy}{dt} = 1 - \sin t - \int_0^t y(\tau) d\tau \quad \text{and} \quad y(0) = 0. \quad (5\%)$$

5. Consider the matrix  $M = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ -1 & 1 & 3 \end{bmatrix}$ .

(a) Find the determinant of  $M$  and obtain the inverse matrix  $M^{-1}$ . (3%+4%)

(b) Estimate the eigenvalues and eigenvectors of  $M$ . (4%+4%)

6. (a) Let  $f(x, y) = \alpha \sin(xy) + x^2 + 4y^2(1 - y)$ . For what values of  $\alpha$  will  $f$  have a local minimum at  $(0, 0)$ ? (5%)

(b) Let  $I(a) = \int_0^1 \frac{x^a - 1}{\ln x} dx$ , ( $a \geq 0$ ). Find  $I'(a)$ . It is evident that  $I(0) = 0$ . Solve  $I(a)$ . (5%)

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考試科目 (代碼)：工程數學 (3101)

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\*請在【答案卷】作答

7. (a) Apply the divergence theorem to evaluate  $\oint_S (\mathbf{F} \cdot \mathbf{n}) dS$  where  $\mathbf{F} = 2r^3 \hat{\mathbf{e}}_r - r \hat{\mathbf{e}}_\theta + 3\theta \hat{\mathbf{e}}_z$  and  $S$  is the surface of the region bounded by the cylinder:  $r \leq 5$ ,  $0 \leq \theta \leq 2\pi$ ,  $0 \leq z \leq 4$ . (5%)

(b) Let  $\mathbf{F} = 3r \hat{\mathbf{e}}_r - 2rz^2 \hat{\mathbf{e}}_\theta + 4r^2 \hat{\mathbf{e}}_z$ . Find  $\nabla \times \mathbf{F}$ . Evaluate  $\oint_C \mathbf{F} \cdot d\mathbf{r}$  where  $C$  is a counterclockwise circle  $x^2 + y^2 = 9$  on the  $xy$  plane. (5%)

[Formula] Divergence and curl in cylindrical coordinates:

$$\nabla \cdot \mathbf{v} = \frac{1}{r} \frac{\partial(rv_r)}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

$$\nabla \times \mathbf{v} = \left( \frac{1}{r} \frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} \right) \hat{\mathbf{e}}_r + \left( \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \right) \hat{\mathbf{e}}_\theta + \frac{1}{r} \left( \frac{\partial(rv_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta} \right) \hat{\mathbf{e}}_z$$

8. (a) Let  $f(x) = \begin{cases} A_0, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$ ,  $A_0 > 0$ .

The Fourier integral representation of  $f(x)$  is  $f(x) = \int_0^\infty [a(\omega) \cos \omega x + b(\omega) \sin \omega x] d\omega$ . Find  $a(\omega)$  and  $b(\omega)$ . (5%)

(b) Consider an infinite beam problem  $Elu'''' + ku = f(x)$  with the loading  $f(x)$  given in (a). The deflection  $u$  can be solved and written as  $u(x) = \int_0^\infty [c(\omega) \cos \omega x + d(\omega) \sin \omega x] d\omega$ . Find  $c(\omega)$  and  $d(\omega)$ . (5%)

9. Solve the Dirichlet problem in polar coordinates:  $\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$ ,  $1 \leq r \leq 2$ ,  $0 \leq \theta \leq \frac{\pi}{4}$ , with  $u(r, 0) = 0$ ,  $u\left(r, \frac{\pi}{4}\right) = 0$ ,  $u(1, \theta) = 0$ ,  $u(2, \theta) = 10$ . (10%)

10. (a) Evaluate  $\oint_C \frac{z-1}{(z+1)(z-2)^2} dz$  where  $C$  is the counterclockwise circle  $|z| = 4$ . (5%)

(b) Evaluate the integral  $\int_0^\infty \frac{\ln x}{x^2+4} dx$  by residue theorem. (5%)