

國立清華大學 107 學年度碩士班考試入學試題

系所班組別：聯合招生 (0598)

考試科目 (代碼)：工程數學 (9801)

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\*請在【答案卷】作答

1. Solve the differential equations and provide general solutions of  $y(x)$ .

(a)  $\frac{dy}{dx} + 3x^2y = x^5$  (5%)

(b)  $(y - x^2y)\frac{dy}{dx} = x + 1$  (5%)

(c)  $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = (\ln x)x^3$  (5%)

2. Use the Laplace transform to solve the problem

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 6y = f(t), \text{ where } f(t) = \begin{cases} 0, & 0 \leq t < 1 \\ 1, & 1 \leq t < 2, \\ 0, & t \geq 2 \end{cases}$$

$y(0) = 1$ , and  $y'(0) = -1$ . (10%)

You may express  $f(t)$  in terms of the unit step function.

3. Find the series solution of the following differential equation about  $x=0$ .

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - \frac{1}{4}\right)y = 0. \quad (10\%)$$

You have to express the solution in the form of  $y(x) = C_1y_1(x) + C_2y_2(x)$ . To save time, you can only show the first three terms of  $y_1(x)$  and  $y_2(x)$ .

4. Suppose the matrix

$$M = \begin{bmatrix} 7 & -2 & -1 \\ 3 & 0 & 1 \\ 9 & -2 & -3 \end{bmatrix}.$$

(a) Find the determinant of  $A$  (3%) and obtain the inverse matrix  $A^{-1}$  (4%).

(b) Estimate the eigenvalues (4%) and eigenvectors of  $A$  (4%).

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5. (a) Evaluate  $\oint_C 3zdx + 2xdy + ydz$ , where  $C$  is the trace of the cylinder  $x^2 + y^2 = 4$  in the plane  $y + z = 6$ . Orient  $C$  counterclockwise as viewed from above. (5%)

(b) Let  $S$  be the surface of the region bounded by the hemisphere  $(x - 3)^2 + (y - 2)^2 + (z - 1)^2 = 25$ ,  $1 \leq z \leq 6$ , and the plane  $z = 1$ .

Evaluate  $\iint_S (\mathbf{F} \cdot \mathbf{n})dS$  where  $\mathbf{F} = x\mathbf{i} + 2y\mathbf{j} - 5(z - 1)\mathbf{k}$ . (5%)

6. Use Fourier transform  $\tilde{u}(k, t) = \int_{-\infty}^{\infty} u(x, t)e^{ikx}dx$  to solve the heat equation

$$\alpha \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad -\infty < x < \infty, \quad t > 0, \quad \text{subject to } u(x, 0) = \begin{cases} u_0, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases} \quad (10\%)$$

7. Solve the nonhomogeneous boundary value problem:  $2 \frac{\partial^2 u}{\partial x^2} + 4 = \frac{\partial u}{\partial t}$  with

$$u(0, t) = 0, u(1, t) = 1 \text{ for } t > 0, \text{ and } u(x, 0) = -x^2 \text{ for } 0 < x < 1.$$

Hint: Assume that  $u(x, t) = v(x, t) + \psi(x)$ . If one sets  $\psi'' + 2 = 0$ , the nonhomogeneous PDE can reduce to a homogeneous PDE. (10%)

8. Evaluate  $\oint_C \frac{1}{(z+1)^2(z-2)}dz$ , where  $C$  is the ellipse  $\frac{x^2}{9} + y^2 = 1$  and the loop orientation is counterclockwise. (10%)

9. A flow in a corner, sketched in the figure (left), satisfies the Laplace equation

$$\nabla^2 \phi = 0 \quad \text{with the boundary condition } \nabla \phi \cdot \hat{\mathbf{n}} \equiv \frac{\partial \phi}{\partial n} = 0 \quad \text{where } \phi(x, y) \text{ is the}$$

scalar velocity potential. Use the conformal mapping  $w = z^2$  to determine  $\phi(x, y)$  and the velocity field  $\mathbf{q} = \nabla \phi$ . Hint: After the conformal mapping, the flow becomes a uniform stream (refer to the figure sketched on the right). We

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assume that the uniform stream has speed  $U_0$ .

(10%)

