國立清華大學 106 學年度碩士班考試入學試題

系所班組別：聯合招生 (0598)

考試科目（代碼）：電磁學 (9803)

　　*請在【答案卷】作答

請注意：1. 請以 MKS 制單位回答問題

2. 電磁常數及常用公式給於所有題目之後

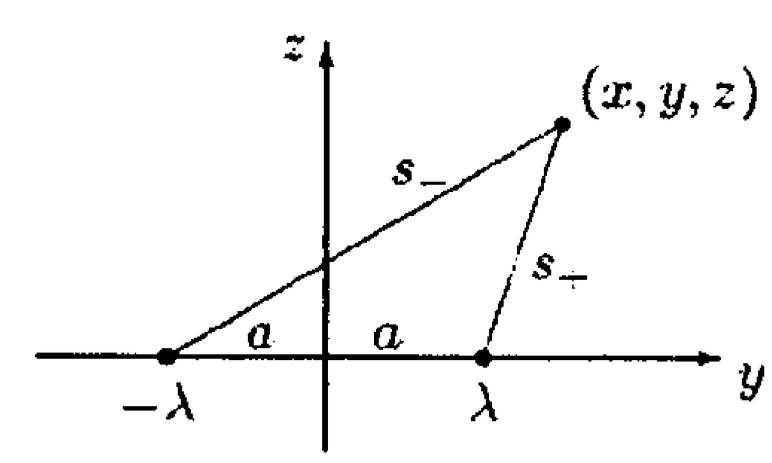
1. (a) (5%) Find the capacitance per unit length of two coaxial metal cylindrical tubes of radii a and b. $(a < b)$

(b) (10%) Find the electric field in the whole space, produced by a uniformly polarized sphere of radius R, assume that the polarization is $\mathbf{P} = P\hat{\mathbf{z}}$. [Hint: the potential produced by a surface charge density $\sigma_0 \cos\theta$ on a sphere of radius R is given by

$$V(r,\theta) = \begin{cases} \frac{\sigma_0}{3\epsilon_0} r\cos\theta, & \text{for } r \le R \\ \frac{\sigma_0 R^3}{3\epsilon_0 r^2}\cos\theta, & \text{for } r \ge R \end{cases}$$

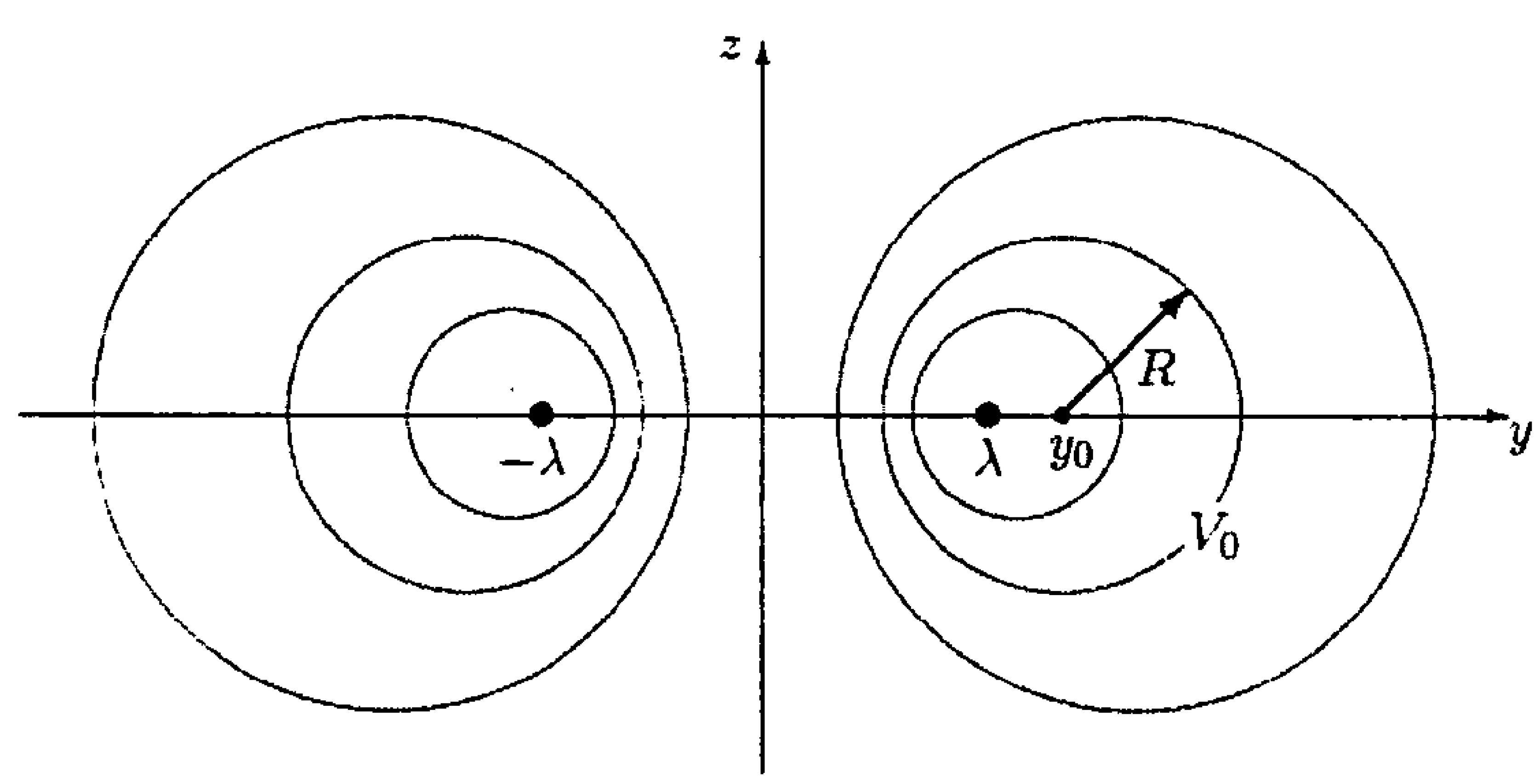
]

2. Two infinite long wires running parallel to the x axis carry uniform charge density $+\lambda$ and $-\lambda$, and are separated by a distance $2a$.



(a) (5%) Find the potential V at any point (x, y, z).

(b) (10%) Show that the equipotential surfaces are circular cylinders. Locate the axis y_0 and radius R of the cylinder corresponding to a given potential V_0.



3. (10%) A sphere shell of radius R, carrying a uniform surface charge σ, is spinning at angular velocity $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$. Calculate the magnetic field $\mathbf{B}$ inside and outside the sphere, given the vector potential produced at any point $\mathbf{r}$ to be

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \frac{\mu_0 R\sigma}{3}(\boldsymbol{\omega}\times\mathbf{r}), & (r \le R) \\ \frac{\mu_0 R^4\sigma}{3r^3}(\boldsymbol{\omega}\times\mathbf{r}), & (r \ge R) \end{cases}$$

4. (a) (10%) A coaxial cable consists of two very long cylindrical tubes of radii a and b (assumed $a < b$), separated by linear insulating material of magnetic susceptibility χ_m. A current I flows down the inner conductor and returns along the outer one. In each tube, the current distributes itself uniformly over the surface. Calculate the magnetization $\mathbf{M}$, the volume bound current density $\mathbf{J}_\mathrm{b}$, the surface bound current density $\mathbf{K}_\mathrm{b}$, and the magnetic field $\mathbf{B}$ in the region between the tubes.

(b) (10%) A sphere of linear magnetic material of susceptibility χ_m is placed in a uniform magnetic field $\mathbf{B}_0$. Find the resulting magnetic field inside the sphere. [Hint: the magnetic field setup by a magnetization $\mathbf{M}$ within a sphere is $\mathbf{B} = \frac{2}{3}\mu_0\mathbf{M}$]

5. (15%) The complex wave number of an electromagnetic wave in materials satisfies the relation $\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega$.

(a) Show that the skin depth in a poor conductor is $(\frac{2}{\sigma})\sqrt{\frac{\epsilon}{\mu}}$.

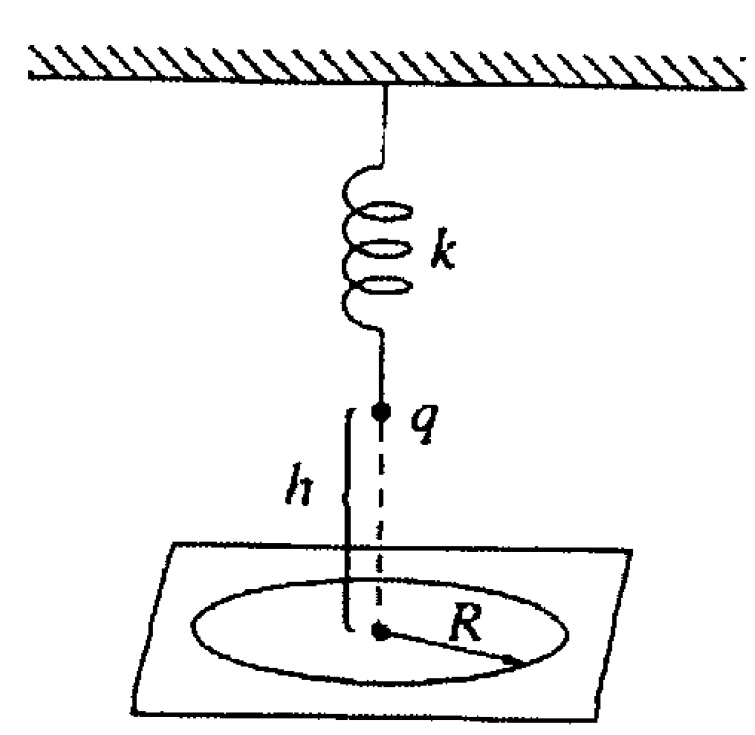
(b) Find the skin depth in water (For water, the permittivity $\epsilon = 80\epsilon_0$, the permeability $\mu \cong \mu_0$, the conductivity $\sigma \cong 5 \times 10^{-3}\mathrm{S/m}$).

(c) Show that the skin depth in a good conductor is $\lambda/2\pi$. (λ is the wavelength.)

(d) Find the skin depth for a typical metal ($\sigma \approx 10^7\mathrm{S/m}$) in the visible range ($\omega \approx 10^{15}/\mathrm{s}$) and $\epsilon \cong \epsilon_0$, $\mu \cong \mu_0$.

6. (10%) Consider the propagation of TE waves in a wave guide of rectangular shape with height a and width b (assume $a \geq b$). The propagation wave vector in the guide can be written as $\mathbf{k}' = k_x\hat{x} + k_y\hat{y} + k\hat{z}$. What are the allowed values for k_x and k_y? What is the cutoff frequency ω_{mn} for each TE mode? Show that the phase velocity of the wave $v = \omega/k$ down the wave guide is greater than light speed c. Find the group velocity v_g.

7. (15%) A particle of mass m and charge q is attached to a spring with force constant k, hanging from the ceiling (see figure). Its equilibrium position is a distance h above the floor. It is pulled down a distance d below equilibrium and



released at $t = 0$. Let c be the light speed, λ be the wavelength of radiation and $\omega = \sqrt{k/m}$.

(a) Under the assumption $d \ll \lambda \ll h$, calculate the intensity of the radiation hitting the floor as a function of R from the point directly below q. [Hint: The average energy radiated by an oscillating electric dipole $P_0 = qd$ is given by the Poynting vector $\langle \mathbf{S} \rangle = \left(\frac{\mu_0 P_0^2 \omega^4}{32\pi^2 c}\right)\frac{\sin^2\theta}{r^2}\hat{\mathbf{r}}$].

(b) Calculate the average total radiation energy per unit time striking the floor of infinite extent. [Hint: Integral: $\int_0^\infty \frac{x^{a-1}}{(x+1)^{a+b}}dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$]

(c) Because it is losing energy in form of radiation, the amplitude of the oscillation gradually decreases. How long will the amplitude be reduced to $d \cdot \exp(-1)$?

電磁常數及常用公式:

Vacuum permittivity $\epsilon_0 = \frac{10^{-9}}{36\pi}$ F/m

Vacuum permeability $\mu_0 = 4\pi \times 10^{-7}$ H/m

Light speed in vacuum $c = 3 \times 10^8$ m/s

Maxwell's equations:

(in general)

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

(in matter)

$$\nabla \cdot \mathbf{D} = \rho_f$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

Auxiliary fields: $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$, $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$

Linear media: $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$, $\mathbf{D} = \epsilon \mathbf{E}$, $\mathbf{M} = \chi_m \mathbf{H}$, $\mathbf{H} = \frac{1}{\mu}\mathbf{B}$

Potentials: $\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$, $\mathbf{B} = \nabla \times \mathbf{A}$

Vector derivatives:

Spherical: $d\mathbf{l} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta\, d\phi\hat{\phi}, \quad dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$

$$\nabla t = \frac{\partial t}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\phi}$$

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial (r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, v_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial\left(\sin\theta\, v_\phi\right)}{\partial\theta} - \frac{\partial v_\theta}{\partial\phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial(r v_\phi)}{\partial r}\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial(r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial\theta}\right]\hat{\phi}$$

$$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

Cylindrical: $d\mathbf{l} = dr\hat{r} + rd\theta\hat{\theta} + dz\hat{z}, \quad dV = r\, dr\, d\theta\, dz$

$$\nabla t = \frac{\partial t}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial t}{\partial\theta}\hat{\theta} + \frac{\partial t}{\partial z}\hat{z}$$

$$\nabla \cdot \mathbf{v} = \frac{1}{r}\frac{\partial(r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{\partial v_z}{\partial z}$$

$$\nabla \times \mathbf{v} = \left[\frac{1}{r}\frac{\partial v_z}{\partial\theta} - \frac{\partial v_\theta}{\partial z}\right]\hat{r} + \left[\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial(r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial\theta}\right]\hat{z}$$

$$\nabla^2 t = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial t}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 t}{\partial\theta^2} + \frac{\partial^2 t}{\partial z^2}$$