國立清華大學 105 學年度碩士班考試入學試題

系所班組別：聯合招生 (0598)

考試科目（代碼）：工程數學 (9801)

　　*請在【答案卷】作答

1. Find the interval of x for which the given ODE or IVP has an unique solution.

(a) $\frac{1}{\ln x}\frac{dy}{dx} - y = 0$ (2%)

(b) $\frac{d^2y}{dx^2} + \tan(x)\, y = e^x,\ y(0) = 1, y'(0) = 0$ (2%)

2. Obtain general solution for following ODE. (6 %)

$$\frac{dy}{dx} = \frac{4x^2 + y^2}{xy}$$

3. Solve $x\frac{d^2y}{dx^2} + (1 - x)\frac{dy}{dx} - y = 0$ about $x = 0$. (11 %)

4. Solve following initial value problem: (8 %)

$$\frac{d^2y}{dt^2} + y = g(t), \qquad y(0) = 0, \qquad y'(0) = 1$$

where $g(t) = \begin{cases} 0, & 0 \le t < \pi \\ 1, & \pi \le t < 2\pi \\ 0, & t \ge 2\pi \end{cases}$

5. Solve following ODE for general solution: (6 %)

$$\frac{d^3x}{dt^3} + \frac{d^2x}{dt^2} - 2x = 0$$

6. (a) Find the the eigenvalues ($\lambda_1 \geq \lambda_2 \geq \lambda_3$) of the matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix}$ (5%)

(b) Obtain an orthogonal matrix P which gives $P^T A P = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$. (5%)

7. The position of a moving particle is given by $\vec{r}(t) = (3\cos t, 3\sin t, 4t)$. Find the unit tangent vector $\vec{T}(t)$, the unit normal vector $\vec{N}(t)$, and the binormal vector $\vec{B}(t)$. Find the curvature $\kappa(t)$. (5%)

8. (a) Evaluate $\oint_C (x^7 + 8y)dx + (6x - e^y)dy$ on the curve $C: (x-5)^2 + (y+3)^2 = 5$, integrating along the positive direction. (5%)

(b) Evaluate $\iint_S (\vec{F} \cdot \vec{n})\, dS$ where $\vec{F} = (2x, 3y^2, 1 - 4z)$ and $\vec{n}$ is the outward normal of the surface S of a cubic bounded by $-1 \leq x \leq 3$, $0 \leq y \leq 4$, $5 \leq z \leq 9$. (5%)

9. (a) Assume that $f(x)$ is a 2π-periodic function, defined in $(-\infty, \infty)$. Given

$f(x) = \begin{cases} 1, & -\frac{\pi}{2} \leq x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \leq x < \frac{3\pi}{2} \end{cases}$, we expand $f(x)$ in Fourier series :

$f(x) = \sum_{n=0}^{\infty} a_n \cos(nx)$. Find a_n $(n = 0, 1, 2, 3, \ldots)$ (5%)

(b) The differential equation $5y'''' + 4y = f(x)$ can be solved by seeking the solution in the form $y(x) = \sum_{n=0}^{\infty} d_n \cos(nx)$, where $f(x)$ is given in (a). Find d_n $(n = 0, 1, 2, 3, \ldots)$ (5%)

10. Let $u(x, t)$ satisfy the equation (15%)

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{for } 0 < x < 1 \text{ and for } t > 0,$$

subject to the initial condition

$$u(x, 0) = f(x), \qquad \text{for } 0 < x < 1,$$

and the boundary conditions

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=1} = -h\,u(1, t), \qquad \text{for } t > 0,$$

where h is a positive constant. Solve for $u(x, t)$. It is required to show details of your work, including the ***derivation*** of the relevant orthogonal set of eigenfunctions to this problem.

11. Use Cauchy's residue theorem to evaluate the integral (15%)

$$\oint_C \frac{e^z}{1-\cos z}\,dz,$$

where C is the *positively oriented* circle $|z| = 1$. It is required to discuss (a) how to locate and ***classify*** the singularity of the function $f(z) = \dfrac{e^z}{1-\cos z}$ within the indicated contour C, and (b) how to find the corresponding residue.