國立清華大學 103 學年度碩士班考試入學試題

系所班組別：聯合招生 (0598)

考試科目（代碼）：工程數學 (9801)

　　*請在【答案卷、卡】作答

1. The ODE $y'+p(t)y=q(t)y^n$ (n is a constant) is called Bernoulli's equation.

(a) Give the general solution for the special cases $n=0$ and $n=1$. (5%)

(b) If n is neither 0 nor 1, the ODE can be solved by transforming $y(t)$ to $u(t)=y^{1-n}(t)$. Find the ODE for $u(t)$. (5%)

(c) Use the suggested method in (b) to solve $2tyy'+y^2=2t$. (10%)

2. Applying Laplace transform on the two sides of the equation $y''+ay'+by=r(t)$ (a,b: constants) with initial conditions $y(0)$ and $y'(0)$, we obtain the subsidiary equation of the ODE: $Y(s)=[y'(0)+y(0)F(s)+R(s)]Q(s)$, where $Y(s)$ and $R(s)$ are the Laplace transform of $y(t)$ and $r(t)$, respectively. $y(t)$ can be then solved by taking the inverse Laplace transform of $Y(s)$.

(a) Determine the functions $F(s)$ and $Q(s)$ in the subsidiary equation. (5%)

(b) Use the method described above and Laplace convolution to solve the equation

$$y''+3y'+2y=r(t),\quad r(t)=\begin{cases}2 & \text{if } 2<t<4\\ 0 & \text{otherwise}\end{cases},\quad y(0)=0,\ y'(0)=0. \quad (10\%)$$

3. Let $[A]=\begin{bmatrix}-3 & 1\\ 1 & -3\end{bmatrix}$ be a matrix.

(a) Determine the eigenvalues and the corresponding eigenvectors of $[A]$. (5%)

(b) Find the general solution of the linear ODE system: $\begin{bmatrix}y_1'\\ y_2'\end{bmatrix}=[A]\begin{bmatrix}y_1\\ y_2\end{bmatrix}$. (5%)

(c) Solve the nonhomogeneous linear ODE system:

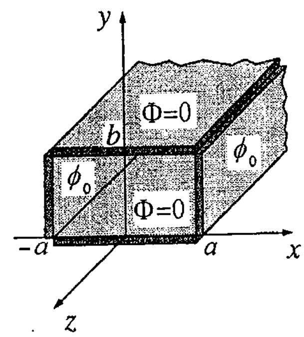
$$\begin{bmatrix}y_1'\\ y_2'\end{bmatrix}=[A]\begin{bmatrix}y_1\\ y_2\end{bmatrix}+\begin{bmatrix}-6\\ 2\end{bmatrix}e^{-2t},\quad y_1(1)=-4e^{-2},\quad y_2(1)=0. \quad (10\%)$$

4. Let $\vec{F}=(x-2y,\ 2y+x,\ 5z)$ be a vector field in the Cartesian space.

(a) Apply the divergence theorem to evaluate the surface integral $\oiint_S \vec{F}\cdot\hat{n}\,dA$ where $S: x^2+y^2+z^2=25$ is the surface and $\hat{n}$ the surface normal pointing outward. (5%)

(b) Evaluate the line integral of $\vec{F}$ along a closed curve $C: x^2+y^2=4,\quad z=-3$ (counter-clockwise as seen by a person standing at the origin) by Stokes' theorem. (5%)

5. In this question, we solve the PDE in a 3D rectangular space,



$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\Phi(x,y,z)=0$ with the boundary conditions:

$\Phi(\pm a,y,z)=\phi_0$, $\Phi(x,0,z)=0$, $\Phi(x,b,z)=0$ for $-\infty<z<\infty$.

(a) Show, by the method of separating variables, that the eigenfunctions of the PDE take the form $u_n(x,y,z)=\cosh(k_n x)\sin(k_n y)$, $n=1,2,3,\ldots$. Determine $k_n=?$ (5%)

(b) The solution can be expressed by $\Phi(x,y,z)=\sum_{n=1}^{\infty}c_n u_n(x,y,z)$. Find $c_n=?$ (5%)

6. Let $f(z)=\dfrac{3z^2}{(z^2+\frac{1}{4})(2-z)}$ be a complex function.

(a) Find the poles of $f(z)$. Determine the order of pole and the residue at each pole. (10%)

(b) Evaluate the integral $\oint_C f(z)\,dz$, $C:|z|=1$ (oriented counterclockwise). (5%)

7. A 2D Laplace equation $\nabla^2\Phi(r,\theta)=0$ on a disk of radius R for Dirichlet boundary condition can be solved by Poisson's integral formula and the solution can be expressed by a series, as

$$\Phi(r,\theta)=a_0+\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n(a_n\cos n\theta+b_n\sin n\theta),$$

where
$$\begin{cases} a_0 = \frac{1}{2\pi}\int_0^{2\pi} \Phi(R,\mu)\, d\mu, \\ a_n = \frac{1}{\pi}\int_0^{2\pi} \Phi(R,\mu)\cos n\mu\, d\mu, \qquad n = 1, 2, 3, \ldots \\ b_n = \frac{1}{\pi}\int_0^{2\pi} \Phi(R,\mu)\sin n\mu\, d\mu \end{cases}$$

Given the boundary condition $\Phi(2,\theta) = \theta$ for $0 \le \theta < 2\pi$, find the solution $\Phi(r,\theta)$ on the disk ($r \le 2$), expanded up to the $n = 2$ term. (10%)