類組：電機類　科目：工程數學C(3005)　※請在答案卷內作答

1. (15 pts). Write down your proof **as detailed as possible.**
Let V be a finite-dimensional vector space over a field F. The dual space V^* of V is defined as the vector space of linear functionals on V, i.e. $V^* = \{f | f : V \to F\}$. Let $T : V \to V$ be a linear operator. T's transpose $T^t : V^* \to V^*$ is a linear mapping from V^* to V^* defined by $T^t(g) = gT$ for each $g \in V^*$. Let $V = P(\mathbb{R})$, the vector space of polynomials over real numbers. For each positive integer k, define $\varphi_k : V \to \mathbb{R}$ by $\varphi_k(f(x)) = f^{(k)}(0)$, the k-th derivative of $f(x)$ at $x = 0$. Let $\partial : V \to V$ be the differentiation mapping defined by $\partial(f(x)) = f'(x)$. Prove that $\partial^t \varphi_k = \varphi_{k+1}$.

2. (15 pts). Let $V = C^\infty$, the vector space of all real functions having derivatives of all orders, and let $y_1, y_2, \ldots, y_n$ be some fixed linearly independent functions in V. Let $\delta : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ be an alternating n-linear function (defined for each $n \times n$ matrix over $\mathbb{R}$) that is not identically to zero. For each $y \in V$ and $t \in \mathbb{R}$, define $T(y(t)) \in \mathbb{R}$ as follows.

$$T(y(t)) = \delta\begin{pmatrix} y(t) & y_1(t) & y_2(t) & \cdots & y_n(t) \\ y'(t) & y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & \vdots & & \vdots \\ y^{(n)}(t) & y_1^{(n)}(t) & y_2^{(n)}(t) & \cdots & y_n^{(n)}(t) \end{pmatrix}$$

(a) Prove that $T : V \to V$ is a linear transformation.

(b) Prove that the null space of T satisifes $N(T) \supset \mathrm{Span}(\{y_1, y_2, \ldots, y_n\})$.

3. (15 pts). **Eigenvalues and eigenvectors.**

(a) Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$. Find its eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$.

(b) Continuing from above, find a matrix $Q \in M_{2\times 2}(\mathbb{C})$ such that $Q^{-1}AQ$ is diagonal.

(c) Find the minimum positive integer n such that $A^n = I$.

4. (10 pts). **Least-square approximation.** Let $f(t)$ be defined as follows,

$$f(t) = \begin{cases} 1, & \text{if } 0 \le t < \pi \\ -1, & \text{if } -\pi < t < 0. \end{cases}$$

Also, define

$$g(t) = a\cos t + b\cos 2t + c\sin t.$$

Find the coefficients (a, b, c) such that $E = \int_{-\pi}^{\pi} |g(t) - f(t)|^2\, dt$ is minimized.

參考用

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5. (25 pts). For a system of non-linear ordinary equations (with $m > 0$) in a two dimension phase plane

$$\begin{aligned} y_1' &= y_2 - 2, \\ y_2' &= \frac{2m}{\pi} y_1 - \sin y_1, \end{aligned}$$

(a) For $m = 1$, please find all the critical points in the phase plane.

(b) Find the range for the value of m such that this system of ordinary differential equation has seven critical points.

6. (20 pts). Derive the *Legendre's equation* from the Laplacian in spherical coordinate, *i.e.*, from the corresponding Laplacian in Spherical coordinates

$$\nabla^2 u = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial u}{\partial \phi}\right) + \frac{1}{\sin^2\phi}\left(\frac{\partial^2 u}{\partial \theta^2}\right)\right] = 0.$$

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