科目：工程數學 C(5005)　　　　校系所組：中央大學電機工程學系(電子組、系統與生醫組)
交通大學電子研究所(甲組、乙組)
交通大學電控工程研究所(甲組、丙組)
交通大學電信工程研究所 (乙組)
清華大學電機工程學系(甲組)
清華大學光電工程研究所
清華大學電子工程研究所
清華大學工程與系統科學系 (丁組)

一． (5%)

(一) (2%) Solve $y' = y^2, y(0) = 2$. Call the solution y_c.

(二) (2%) Solve $y' = y^2 - 1, y(0) = 1$. Call the solution y_p.

(三) (1%) Does $y_c + y_p$ solve $y' = y^2 - 1, y(0) = 3$? Explain.

二． (8%) One solution of the equation $y'' + p(t)y' + q(t)y = 0$ is $(1+t)^2$, and the Wronskian of any two solutions is constant. Find the general solution of $y'' + p(t)y' + q(t)y = 1 + t$.

三． (5%) Three solutions of a 2nd-order linear equation $L(y) = g(t)$ are

$$\psi_1 = 2e^{t^2} + e^t, \psi_2 = te^{t^2} + e^t \ and \ \psi_3 = (1+t)e^{t^2} + e^t.$$

Find the solution of the initial problem $L(y) = g(t); y(0) = 3, y'(0) = 0$.

四． (8%) Let y be a real function of x. Find two linearly independent Frobenius solutions of the following differential equation at $x = 0$:

$$2x^2 y'' + x(x-3)y' + 3y = 0$$

五． (8%) Let x_1 and x_2 be two real functions of t. Solve x_1 and x_2 for the following system of differential equations

$$\begin{cases} x_1' = 4x_1 - x_2 \\ x_2' = x_1 + 2x_2 \end{cases}, x_1(1) = 5, x_2(1) = 3$$

六． (7%) Given the initial value problem, $x'' + 4x' + 13x = f(t); x(0) = x'(0) = 0$

(一) (3%) Express $x(t)$ in terms of $f(t)$ and convolution.

(二) (4%) Solve $x(t)$ for $f(t) = u(t) - u(t-1)$, where $u(t)$ denotes the unit step (or Heaviside Step) function.

注意：背面有試題

七. (9%) $f(t) = \begin{cases} 1, & 0 < t < 5 \\ 0, & 5 < t < 10 \end{cases}$ with $f(t+10) = f(t)$ is a piecewise continuous and periodic function that

satisfies $f(t) = \dfrac{|f(t^+) + f(t^-)|}{2}$, where $f(t^+)$ and $f(t^-)$ are the right-hand and left-hand limits of $f(t)$ at each

discontinuity.

(一) (3%) Find the Fourier series of $f(t)$.

(二) (3%) Let $f(t)$ be defined for $t \geq 0$; find its Laplace transform $F(s)$ for $s > 0$.

(三) (3%) Find a particular solution for $x'' + 16x = f(t)$.

八. (9%) Consider the following partial differential equation: $\dfrac{\partial^2 u}{\partial t^2} = 4\dfrac{\partial^2 u}{\partial x^2}$

For the moment let's assume that u is in the form of $u = A\sin\lambda x \cos\omega t$, where A, λ, and ω are non-zero constants.

(一) (3%) What is the relationship between λ and ω?

(二) (3%) Assuming that u must satisfy the condition $u(x=0, t) = u(x=8, t) = 0$, what is λ? Notice that λ might not be single valued.

(三) (3%) What is the lowest possible frequency of u?

九. (4%) Given the equation for the circular membrane vibration: $\dfrac{\partial^2 u}{\partial t^2} = c^2\left(\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r}\right)$

with the boundary condition: $u(r=20, t) = 0$ and c is a constant, the solution is in the form of

$$u = \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} W_n T_n = \sum_{n=1}^{\infty}(A_n \cos ck_n t + B_n \sin ck_n t)J_0(k_n r)$$

where J_0 is the Bessel function of the first kind. Find the nodal line (curves on the membrane that do not move) for $n = 2$. Note that J_0 has infinitely many positive zeros α_1, α_2, α_3 ... as shown in the following figure.
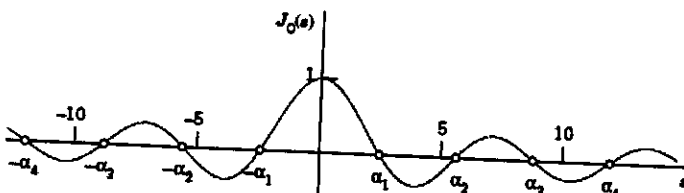
十. (12%) Consider a bulk of silicon with dimension $l \times w \times d$, where the thickness d (along the z-axis) is much smaller than the length l and width w. The diffusion and annealing of dopant ions is governed by the diffusion equation:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial z^2}, \quad (t > 0, \; 0 < z < d)$$

where $u(z,t)$ means the dopant ion density and α^2 represents the diffusivity.

(一)(6%) Let the dopant ion density profile somewhere inside the silicon at a certain instant t_0, i.e. $u(z,t_0)$, look like as shown in the figure. Without detailed calculation, please roughly draw the new dopant ion density profile corresponding to a very short time Δt later, i.e. $u(z,t_0 + \Delta t)$.
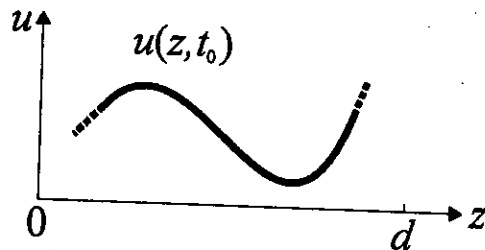
(二)(3%) Write down the boundary conditions if all the dopant ions cannot escape from the bulk of silicon.

(三)(3%) Without detailed calculation, can you draw the steady-state dopant ion density profile $u(z,t \to \infty)$ by using $u(z,t_0)$ as shown and the boundary conditions specified in part (二)?

十一. (13%) For a 3x3 matrix A, where

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$$

(一) (6%) Please find its eigenvalues and corresponding eigenvectors.

(二) (7%) Assume the 3 eigenvalues are in the order of $\lambda_1 \leq \lambda_2 \leq \lambda_3$. Starting from the eigenvector corresponding to λ_1, please find the corresponding orthonormal basis.

十二. (12%) Define the space P_n as the set of all polynomials of degree less than n. Let L be the operator on P_3 and

$$L(p(x)) = \alpha + x\frac{d}{dx}p(x) + \frac{d^2}{dx^2}p(x)$$

(一) (3%) Find the matrix $\mathbf{A}$ representing L with respect to $[1, x, x^2]$.

(二) (3%) Find the matrix $\mathbf{B}$ representing L with respect to $[1, x, 1+x^2]$.

(三) (3%) Find the condition of α such that $\mathbf{A}$ and $\mathbf{B}$ are similar matrices.

(四) (3%) If $p(x) = a_0 + a_1 x + a_2 x^2$, calculate $L^n(p(x))$ given the condition of α in (三).