※請在答案卡內作答

- 本測驗試題為複選題（答案可能有一個或多個），請選出所有正確或最適當的答案，並請用2B鉛筆作答於答案卡。
- 共二十題，每題完全答對得五分，答錯不倒扣。

Notation: In the following questions, underlined letters such as $\underline{a}, \underline{b}$, etc. denote column vectors of proper length; boldface letters such as $\mathbf{A}, \mathbf{B}$, etc. denote matrices of proper size; $\mathbf{A}^\top$ means the transpose of matrix $\mathbf{A}$. $\mathbb{R}$ is the usual set of all real numbers. By $\mathbf{A} \in \mathbb{R}^{m\times n}$ we mean $\mathbf{A}$ is an $m \times n$ real-valued matrix. $u(x)$ is unit-step function defined as $u(x) = 1$ if $x \geq 0$ and $u(x) = 0$ if $x < 0$; $\star$ is the convolution operator; $\mathscr{L} : f(x) \mapsto F(s)$ and $\mathscr{L}^{-1} : F(s) \mapsto f(x)$ denote the unilateral Laplace and inverse Laplace transforms for $x \geq 0$, respectively.

一、To solve a system of linear equations, we often use the augmented matrix form $[\mathbf{A}\ \underline{b}]$; then reduce $\mathbf{A}$ to an upper triangular matrix by operating on the rows of $\mathbf{A}$ and carry out the same operations on the vector $\underline{b}$. For instance, consider

$$\begin{cases} x + 2y + 2z &= 1 \\ 4x + 8y + 9z &= 3 \\ 3y + 2z &= 1 \end{cases} \tag{1}$$

so we have

$$\begin{bmatrix} \mathbf{A} & \underline{b} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 4 & 8 & 9 & 3 \\ 0 & 3 & 2 & 1 \end{bmatrix}.$$

Then we can apply an elimination matrix $\mathbf{E}$ to the system such that $\mathbf{E}[\mathbf{A}\ \underline{b}] = [\mathbf{U}\ \underline{c}]$ and $\mathbf{U}$ is an upper triangular matrix. Which of the following statements are true?

(A) Left-multiply matrix $\mathbf{A}$ by $\mathbf{E}_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ to subtract four times the first row of $\mathbf{A}$ from the second row of $\mathbf{A}$.

(B) Left-multiply matrix $\mathbf{A}$ by $\mathbf{P}_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ to exchange the second and the third columns of $\mathbf{A}$.

(C) We can do both steps at once, i.e., set $\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{U} = \mathbf{EA}$ is an upper triangular matrix.

(D) The solution to the linear system (1) is $x = 1$, $y = 1$ and $z = -1$.

(E) None of the above is true.

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二、Suppose that matrix $\mathbf{A}$ can be factorized into the following form

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 & 5 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Which of the following statements are true?

(A) $\text{rank}(\mathbf{A}) = 2$

(B) The dimension of the row-space of $\mathbf{A}$ equals 3.

(C) The dimension of the column-space of $\mathbf{A}$ equals 2.

(D) The dimension of the nullspace of $\mathbf{A}$ equals 1.

(E) None of the above is true.

三、Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{4\times4}$ be matrices. Suppose that $\mathbf{A}$ and $\mathbf{B}$ have the same column space, but may not have the same columns. Which of the following statements are true?

(A) Both matrices $\mathbf{A}$ and $\mathbf{B}$ have the same number of pivots.

(B) Both matrices $\mathbf{A}$ and $\mathbf{B}$ have the same left nullspace and the same nullspace.

(C) If $\mathbf{A}$ is invertible, so is $\mathbf{B}$.

(D) The row spaces of matrices $\mathbf{A}$ and $\mathbf{B}$ are orthogonal to each other under the usual Euclidean inner product.

(E) None of the above is true.

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四、 Consider the matrix equation $\mathbf{A}\underline{x} = \underline{b}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \end{bmatrix} \quad \text{and } \underline{b} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Let $\mathbf{P}$ be an orthogonal projection matrix from $\mathbb{R}^3$ onto the column space of $\mathbf{A}$. Which of the following statements are true?

(A) The matrix $\mathbf{A}$ is rectangular. It has no multiplicative left-inverse.

(B) The solution to $\mathbf{A}\hat{\underline{x}} = \mathbf{P}\underline{b}$ is $\hat{\underline{x}} = [1 \;\; 1]^\top$.

(C) The projection vector $\mathbf{P}\underline{b} = \frac{1}{3}[2 \;\; 3]^\top$.

(D) $\mathbf{P} = \mathbf{A}\left(\mathbf{A}^\top\mathbf{A}\right)^{-1}\mathbf{A}^\top$; $\mathbf{P}$ is symmetric and satisfies $\mathbf{P}^2 = \mathbf{P}$.

(E) None of the above is true.

五、 For the 4×4 matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 3 \\ 1 & 3 & 1 & 9 \end{bmatrix}$$

which of the following statements are true?

(A) $\det(\mathbf{A}) = 3$

(B) $\mathbf{A}$ is invertible

(C) $\det(\mathbf{A}^{-1}) = 3$

(D) The (1,2)th entry of $\mathbf{A}^{-1}$ equals 3.

(E) None of the above is true.

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六、 What are the coordinate vectors for $f(x) = 5x^2 + 2x - 3$, with ordered bases $\mathcal{B}_1 = \{x^2, x, 1\}$ and $\mathcal{B}_2 = \{x^2 - x + 1, 3x^2 + 1, 2x^2 + x - 2\}$, respectively?

(A) $[f(x)]_{\mathcal{B}_1} = \begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix}, [f(x)]_{\mathcal{B}_2} = \begin{bmatrix} 0 \\ -\frac{13}{3} \\ \frac{5}{3} \end{bmatrix}.$

(B) $[f(x)]_{\mathcal{B}_1} = \begin{bmatrix} 0 \\ -\frac{13}{3} \\ \frac{5}{3} \end{bmatrix}, [f(x)]_{\mathcal{B}_2} = \begin{bmatrix} 5 \\ 2 \\ -3 \end{bmatrix}.$

(C) $[f(x)]_{\mathcal{B}_1} = \begin{bmatrix} \frac{2}{3} \\ -\frac{13}{3} \\ 0 \end{bmatrix}, [f(x)]_{\mathcal{B}_2} = \begin{bmatrix} 5 \\ 2 \\ -3 \end{bmatrix}.$

(D) $[f(x)]_{\mathcal{B}_1} = \begin{bmatrix} \frac{2}{3} \\ -\frac{13}{3} \\ 0 \end{bmatrix}, [f(x)]_{\mathcal{B}_2} = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}.$

(E) None of the above is true.

七、 Given a non-zero matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$, which of the following statements are true?

(A) The nullspace of $\mathbf{A}$ equals the left nullspace of $\mathbf{A}$.

(B) The eigenvectors of $\mathbf{A}$ are linearly independent.

(C) Part of the eigen-space can equal the left nullspace of $\mathbf{A}$.

(D) The dimension of the eigen-space of $\mathbf{A}$ equals n.

(E) None of the above is true.

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八、Let T be a linear operator on $\mathbb{R}^3$ defined as

$$T(\underline{x}) = \begin{bmatrix} 2x_2 + x_3 \\ x_1 - 4x_2 \\ 3x_1 \end{bmatrix}, \text{ with } \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

What is the standard matrix, $[T]_{\mathcal{B}}$, associated to T with respect to the ordered basis

$$\mathcal{B} = \left\{ \underline{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \underline{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\},$$

and $[T(\underline{x})]_{\mathcal{B}}$?

(A) $[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 2 & 3 \\ 1 & -4 & 0 \\ 3 & 0 & 0 \end{bmatrix}, [T(\underline{x})]_{\mathcal{B}} = \begin{bmatrix} x_3 \\ x_2 - x_3 \\ x_1 - x_2 \end{bmatrix}.$

(B) $[T]_{\mathcal{B}} = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}, [T(\underline{x})]_{\mathcal{B}} = \begin{bmatrix} 3x_1 \\ -2x_1 - 4x_2 \\ -x_1 + 6x_2 + x_3 \end{bmatrix}.$

(C) $[T]_{\mathcal{B}} = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}, [T(\underline{x})]_{\mathcal{B}} = \begin{bmatrix} x_3 \\ x_2 - x_3 \\ x_1 - x_2 \end{bmatrix}.$

(D) $[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 2 & 3 \\ 1 & -4 & 0 \\ 3 & 0 & 0 \end{bmatrix}, [T(\underline{x})]_{\mathcal{B}} = \begin{bmatrix} 3x_1 \\ -2x_1 - 4x_2 \\ -x_1 + 6x_2 + x_3 \end{bmatrix}.$

(E) None of the above is true.

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九、Let $\mathbf{A} \in \mathbb{R}^{\mathbf{n}\times\mathbf{n}}$ be a non-zero, symmetric matrix. Suppose the eigen-decomposition of $\mathbf{A} = \mathbf{Q}\Lambda\mathbf{Q}^\top$, where Λ is a diagonal matrix and $\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \end{bmatrix}$ is an orthonormal matrix, with $\mathbf{Q}_1 \in \mathbb{R}^{n\times r}$ and $\mathbf{Q}_2 \in \mathbb{R}^{n\times(n-r)}$, $r = \text{rank}(\mathbf{A})$. Which of the following statements are true?

(A) $\text{span}\left(\underline{x} - \mathbf{Q}_1\mathbf{Q}_1^\top\underline{x}\right) = \text{span}\left(\mathbf{Q}_2\mathbf{Q}_2^\top\underline{x}\right)$ for any given vector $\underline{x} \in \mathbb{R}^n$.

(B) $\mathbf{A}\mathbf{Q}_2$ is <u>not</u> an all-zero matrix.

(C) $\mathbf{Q}_1^\top\mathbf{Q}_2$ is an all-zero matrix.

(D) $\mathbf{Q}_2^\top\mathbf{Q}_2$ is an all-zero matrix.

(E) None of the above is true.

十、Which of the following statements are true?

(A) A real matrix may be positive definite without being symmetric with respect to the real Euclidean inner product.

(B) A positive semidefinite matrix cannot have 0 as an eigenvalue.

(C) $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ is a linear transformation.

(D) Let $\mathbf{A}$ and $\mathbf{B}$ be square matrices; then $\det(\mathbf{A}^\top\mathbf{B}) = \det(\mathbf{A})\det(\mathbf{B})$.

(E) None of the above is true.

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十一、 Consider the following non-homogeneous system:

$$\begin{cases} y_1'(x) = 4y_1(x) + 2y_2(x) - 15xe^{-2x} \\ y_2'(x) = 3y_1(x) - y_2(x) - 4xe^{-2x} \end{cases}$$

satisfying $y_1(0) = 7$ and $y_2(0) = 3$. Which of the following statements are true?

(A) $y_1(x) = (6 + 2x - 7x^2)e^{-2x} + e^{5x}$ and $y_2(x) = (-4 + 21x^2)e^{-2x} + 7e^{5x}$.

(B) $y_1(x) = \frac{14+13x^2}{14}e^{-2x} + \frac{42+4x}{7}e^{5x}$ and $y_2(x) = \frac{28+5x}{14}e^{-2x} + \frac{7+10x+6x^2}{7}e^{5x}$.

(C) $y_1(1) = \frac{27}{14}e^{-2} + \frac{46}{7}e^{5}$ and $y_2(1) = \frac{31}{14}e^{-2} + \frac{23}{7}e^{5}$.

(D) $y_1(-1) = -3e^{2} + e^{-5}$ and $y_2(-1) = 17e^{2} + 7e^{-5}$.

(E) None of the above is true.

十二、 Consider the differential equation $(2xy(x) - x^2 - y^2(x) - 1)dx + dy(x) = 0$. Which of the following statements are true?

(A) The differential equation is exact.

(B) $x + y(x) = 0$ is an implicit solution.

(C) If $y(0) = \frac{1}{2}$, then $y(1) = 2$.

(D) If $y(0) = 1$, then $y(2) = 1$.

(E) None of the above is true.

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十三、 Consider the differential equation $xdy(x) - 2y(x)[2x^2 - \ln y(x)]dx = 0$. Which of the following statements are true?

(A) The differential equation is linear.

(B) $y(x) = e^{x^2}$ is a particular solution.

(C) If $y(1) = 1$, then $y(2) = \frac{15}{4}$.

(D) If $y(1) = 1$, then $y'(1) = 4$.

(E) None of the above is true.

十四、 Consider the differential equation $(1 - x^2)y''(x) - 2xy'(x) + 2y(x) = 0$. Which of the following statements are true?

(A) $y(x) = \ln \frac{1+x}{1-x}$ is a particular solution.

(B) $y(x) = x$ is a particular solution.

(C) If $y(0) = 1$ and $y'(0) = 1$, then $y(\frac{1}{2}) = \frac{3}{2} - \frac{\ln 3}{4}$.

(D) If $y(0) = 2$ and $y'(0) = 0$, then $y(\frac{1}{2}) = 2 + \ln 3$.

(E) None of the above is true.

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十五、 Consider the following differential equation:

$$(x^3+1)y^{(3)}(x)+(3-x^2)y''(x)+(4+2x)y'(x)+10y(x) = 4+12x+(4x-8x^3)\cos(2x)+(4x^2-2)\sin(2x)$$

satisfying $y(\pi)=\pi$, $y'(\pi)=3$ and $y''(\pi)=0$. Which of the following statements are true?

(A) $y(0)=0$.

(B) $y(\frac{\pi}{2})=\frac{\pi}{2}$.

(C) $y'(\frac{\pi}{4})=1$.

(D) $y'(1)=\pi$.

(E) None of the above are true.

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十六、 Consider the system of differential equations $\underline{y}'(x) = \mathbf{A}\underline{y}(x) + \underline{f}(x)$, where

$$\mathbf{A} = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix},\ \underline{y}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \text{and } \underline{f}(x) = \begin{bmatrix} x^2 \\ x \end{bmatrix}.$$

Which of the following statements are true?

(A) $e^{\mathbf{A}x}\underline{y}(0) = \begin{bmatrix} 1-3x \\ 1-x \end{bmatrix}$

(B) $\mathscr{L}\{\underline{y}(x)\} = \frac{1}{s^5}\begin{bmatrix} s^4 - 2s^3 + 2s + 4 \\ s^4 - s^3 + s^2 - 2s + 2 \end{bmatrix}$

(C) $\underline{y}(1) = \left[-\frac{7}{6}\ \frac{1}{4}\right]^\top$

(D) $[2\ 1]^\top$ is a generalized eigenvector for matrix $\mathbf{A}$ based on eigenvector $[1\ 0]^\top$.

(E) None of the above is true.

十七、 Let $g(x)$ be a real-valued function given as below

$$g(x) = \begin{cases} e^{-|x|}, & \text{if } x \in (-\pi, \pi), \\ 0, & \text{otherwise} \end{cases}$$

which has a Fourier series (F.S.) representation for $x \in (-\pi, \pi)$ as follows

$$g(x) \stackrel{\text{F.S.}}{=} g_0 + \sum_{n \geq 1} [a_n \cos(nx) + b_n \sin(nx)]$$

Which of the following statements are true?

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(A) $g_0 = \frac{1-e^{-\pi}}{\pi}$

(B) $a_1 = \frac{1-e^{-\pi}}{\pi}$

(C) $b_1 = 0$

(D) $2g_0^2 + \sum_{n\geq 1}(a_n^2 + b_n^2) = \frac{1-e^{-2\pi}}{\pi}$

(E) None of the above is true.

十八、Continued from Question 十七, solve for $y(x)$ the following differential equation

$$y''(x) - 9y(x) = \sum_{n=-\infty}^{\infty} g(x - 2n\pi).$$

for all $x \in \mathbb{R}$ and $x \neq \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$ With $y(0) = y'(0) = 0$, it is already known that the solution $y(x)$ takes the following form

$$y(x) \overset{\text{F.S.}}{=\!=} y_0 + \sum_{n\geq 1} [c_n \cos(nx) + d_n \sin(nx)]$$

Which of the following statements are true?

(A) $y_0 = -\frac{1-e^{-\pi}}{9\pi}$

(B) $c_n < 0$ for all $n \geq 1$

(C) For $n \geq 1$, we have $\frac{|a_n|}{|c_n|} = n^2 + 9$.

(D) With the same initial conditions, the solution $y(x)$ can be alternatively represented as

$$y(x) = f(x) \star \left(\sum_{n=-\infty}^{\infty} \delta(x - 2n\pi) \right)$$

where

$$f(x)=\left[\frac{1}{8}\cosh(3x)-\frac{2u(x)-1}{24}\sinh(3x)-\frac{1}{8}e^{-|x|}\right](u(x+\pi)-u(x-\pi))$$

(E) None of the above is true.

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十九、 Solve the following boundary value problem for the real-valued bi-variate function $f(x,y)$ for $0 \le x \le 2\pi$ and $y \ge 0$ satisfying

$$\frac{\partial^2 f(x,y)}{\partial x^2} = 4\frac{\partial^2 f(x,y)}{\partial y^2}$$

$$f(0,y) = f(2\pi,y) = f(x,0) = 0,\ \left.\frac{\partial f(x,y)}{\partial y}\right|_{y=0} = [\sin(x)]^3.$$

Which of the following statements are true?

(A) $f\left(\frac{\pi}{2},\pi\right) = \frac{4}{3}$.

(B) $f\left(\frac{\pi}{2},\frac{\pi}{2}\right) = -\frac{5\sqrt{2}}{6}$.

(C) $\left.\frac{\partial f(x,y)}{\partial y}\right|_{x=y=\frac{\pi}{2}} = \frac{\sqrt{2}}{4}$

(D) $\left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{x=y=\frac{\pi}{2}} = \frac{3\sqrt{2}}{8}$

(E) None of the above is true.

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二十、 Continued from Question 十九, if we change the boundary value problem to

$$\frac{\partial^2 f(x,y)}{\partial x^2} + 4\frac{\partial^2 f(x,y)}{\partial y^2} = 0$$

while keeping all the boundary- and initial conditions the same for $f(x,y)$, which of the following statements are true?

(A) $\frac{\partial f(x,y)}{\partial y}\Big|_{x=\frac{\pi}{4},y=0} = \frac{\sqrt{2}}{2}$

(B) $\frac{\partial f(x,y)}{\partial y}\Big|_{x=\frac{\pi}{2},y=0} = 1$

(C) $\frac{\partial^2 f(x,y)}{\partial x^2}\Big|_{y=0} = 1$

(D) $\lim_{y\to\infty} \frac{\ln(|f(\frac{3\pi}{4},y)|)}{y} = \frac{3}{2}$.

(E) None of the above is true.

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