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Note: Detailed derivations are required to obtain a full score for each problem.

1. (10%) Let $A = \begin{pmatrix} 0 & 3 & 2 & 1 & -4 \\ 2 & 10 & 10 & 16 & 14 \\ -3 & 0 & -5 & -2 & -7 \\ -2 & -1 & -4 & -3 & -6 \\ 2 & 7 & 8 & 11 & 10 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}$.
   - (a) (3%) Compute rank(A).
   - (b) (2%) Compute rank(AB).
   - (c) (3%) Compute rank(A^tAAA^t).
   - (d) (2%) Compute dim(N(B^tA)).

2. (10%) Let V be the vector space spanned by the ordered basis functions $\beta = \{xe^{ax}, e^{ax}, e^{bx}\}$ where $a, b \in R$ and $a \neq b$. Define a linear transformation $\mathsf{T} : \mathsf{V} \to \mathsf{V}$ with parameters $p, q \in R$:
$$\mathsf{T}(y(x)) = y'' + py' + qy.$$
   - (a) (4%) Find the matrix representation for $[\mathsf{T}]_\beta$.
   - (b) (6%) There are two conditions for p and q such that $\dim(\mathsf{N}(\mathsf{T})) = 2$. For each condition, express p and q in terms of a and b, and also find the corresponding null space.

3. (5%) Let A and B are $n \times n$ square matrices such that $AB = C$ where C is an upper triangular matrix with $C_{ij} \neq 0$ whenever $j \geq i$. Prove that A and B are both invertible.

4. (16%) Let V be a vector space over a field $\mathbb{F}$, T be a linear operator on V, and W be a subspace of V. We say that W is invariant under T if for each vector v in W the vector Tv is also in W. Let W be an invariant subspace for T, and $v \in$ V. The **T-conductor of** v **into** W, denoted by $\mathsf{S_T}(v, \mathsf{W})$, is defined as the set of all polynomials $g(x)$ over $\mathbb{F}$ such that $g(\mathsf{T})v$ is in W, i.e., $\mathsf{S_T}(v, \mathsf{W}) = \{g(x) \in \mathbb{F}[x] | g(\mathsf{T})v \in \mathsf{W}\}$.
   - (a) (8%) Prove the following statement. If W is an invariant subspace for T, then, for each polynomial $g(x) \in \mathbb{F}[x]$, W is invariant under $g(\mathsf{T})$.
   - (b) (8%) Prove that if W is an invariant subspace for T then $\mathsf{S_T}(v, \mathsf{W})$ is a subspace of $\mathbb{F}[x]$, the set of polynomials over $\mathbb{F}$.

5. (9%) Let T be a linear operator on a finite-dimensional inner product space V. Prove that $\mathsf{N}(\mathsf{T^*T}) = \mathsf{N}(\mathsf{T})$, where N(T) is the null space for T.

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6. (5%) Solve $y'' + 5y' + 4y = 10e^{-3x}$

7. (5%) Solve the following differential equation with the initial conditions by using Laplace transform. $u(t)$ is the unit step function.

$y'' - 2y' + y = (e^t + t)u(t)$
$y(0) = 1$
$y'(0) = 0$

8. (5%) Find the eigenvalues and eigenfunctions.

$y'' + \lambda y = 0$; $y(0) = y(\pi/2) = 0$

9. (5%) Find the general solution of $(1 - x^2)y'' - 2xy' + 12y = 0$ using "series solution" when $-1 < x < 1$.

10. (5%) Find the Fourier Series of $f(t)$.

$$f(t) = \begin{cases} 0, & -\pi < t < 0 \\ 1, & 0 < t < \pi \end{cases}$$

$f(t + n2\pi) = f(t)$

11. (5%) Find the principle value of $(3 + 4i)^{1/3}$.

12. (5%) Find the open disk of convergence of the following power series and its radius

$$\sum_{n=0}^{\infty} \frac{n^3}{4^n} (z + 3i)^{3n}$$

13. (5%) Evaluate the integration of

$$\int_C \mathcal{R}e(z)dz$$

where C is the shortest path from $1+i$ to $6+6i$

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14. (10%) Evaluate the integration of

$$\oint_C \frac{\sinh z}{\sin z} dz$$

where $C: |z| = \frac{4}{3}\pi$, clockwise.