國 立 清 華 大 學 命 題 紙

97 學年度 動力機械工程 系（所） 甲、丙、丁 組碩士班入學考試

科目 工程數學 科目代碼 1003 (1103、1203) 共 2 頁第 1 頁 *請在【答案卷卡】內作答

1. Find a general solution of

(i) $\dfrac{dy}{dx} - xy = \dfrac{x}{y}$ (10 %)

(ii) $\dfrac{d^2y}{dx^2} + 4y = \sec 2x$ (10 %)

2. Solve the following initial value problem

$$\frac{d^2y}{dt^2} + 4y = \begin{cases} 0 & 0 < t < \pi \\ 1 & \pi < t < 2\pi \\ 0 & t > 2\pi \end{cases}$$

with $y = 0$ and $\dfrac{dy}{dt} = 2$ at $t = 0$. (10 %)

3. Determine clearly all the nature (real symmetric, anti-symmetric, Hermitian, orthogonal or unitary) of the following matrices. (Each matrix may contain more than one nature. No proof is needed.) (10 %)

(A) $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$; (B) $\begin{bmatrix} 2 & 1-i \\ 1+i & 5 \end{bmatrix}$; (C) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$; (D) $\begin{bmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ -i/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$;

(E) The matrix = $\begin{bmatrix} 1-\cos\theta & -\sin\theta \\ \sin\theta & 1-\cos\theta \end{bmatrix}\begin{bmatrix} 1+\cos\theta & \sin\theta \\ -\sin\theta & 1+\cos\theta \end{bmatrix}^{-1}$

4. Let $\phi(x, y, z)$ and $\varphi(x, y, z)$ be continuous with continuous first and second partial derivatives on a smooth closed surface Σ and its interior M. Suppose both $\nabla\phi = \vec{0}$ and $\nabla\varphi = \vec{0}$ in M. Prove that $\iiint_M \left(\phi\nabla^2\varphi - \varphi\nabla^2\phi\right)dV = 0$. (hint: Gauss's divergence theorem) (10%)

5. Find the complex Fourier integral of $f(x) = x\exp(-|x|)$. (10 %)

6. (i) Show that the following partial differential equation

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + A\frac{\partial u}{\partial x} + Bu\right) \quad \text{where k, A and B are constants}$$

can be transformed into a simplified equation like $\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2}$ by choosing α and β appropriately and letting $u = v\exp(\alpha x + \beta t)$. (10 %)

(ii) Use the previous idea to solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 4\frac{\partial u}{\partial x} + 2u$$

$$u(0, t) = u(\pi, t) = 0 \quad \text{for } t \geq 0$$

$$u(x, 0) = x(\pi - x) \quad \text{for } 0 \leq x \leq \pi$$

(10 %)

7. (i) Please use contour integration to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^2(2+2x+x^2)}dx.$$

(10 %)

(ii) Indicate true or false for each of the following statements about complex variables. (No proof is needed. The wrong answer will be given no score but will be deducted 2 points. 每小題答錯倒扣 2 分) (10 %)

(A) The value of $\lim_{z\to 0}\frac{\bar{z}}{z}$ does not exist;

(B) If $f(z) = xy^2 + ix^2y$, then $\frac{df(z)}{dz}$ and $f(z)$ are analytic at $z = 0$;

(C) $f(z) = (e^x\cos y) + i(e^x\sin y)$ is an analytic function;

(D) If $f(z)$ is analytic, then $\int_{z_1}^{z_2} f(z)dz = -\int_{z_2}^{z_1} f(z)dz$;

(E) A unit disk in z plane is mapping onto the upper half of w plane via the transformation $w = \frac{i-z}{i+z}$.