1. (10%)Consider the following equation

$$A\dddot{x}(t)+B\ddot{x}(t)+C\dot{x}(t)+Dx(t)=f(t)$$

with $\ddot{x}(0)=b,\ \dot{x}(0)=c,\ x(0)=d$ and $f(0)=0$. Knowing that the solution for the above equation be

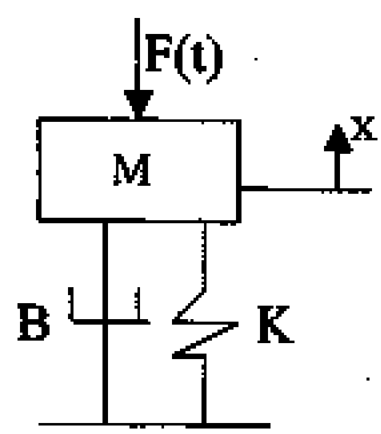
$$x(t)=4e^{-t}+5e^{-3t}-2e^{-7t}+2\sin t-x^2-x,$$

please solve for y(t) in the following equation

$$A\dddot{y}(t)+B\ddot{y}(t)+C\dot{y}(t)+Dy(t)=4f(t)$$

step by step with the initial conditions $\ddot{y}(0)=3b,\ \dot{y}(0)=3c,$ and $y(0)=3d$.

2. (15%)Consider the mechanical system shown below



where the mass is M=1, the damping ratio is B=0.5, the spring constant is K=10, and $x=0$ is the equilibrium position of the system. Assume that the external load is $F(t)=10\cos t$ downward, the initial displacement of the mass is $x(0)$= -2, and the initial velocity of the mass is $\dot{x}(t)$=5, please solve for the position of the mass for all time t.

3. (15%)Let $x(t)$ be the solution of the following equations

$$3\ddot{x}(t)+x(t)+\ddot{y}(t)+3y(t)=e^{t}$$
$$2\ddot{x}(t)+x(t)+\ddot{y}(t)+2y(t)=e^{-t}$$

with the initial conditions x(0)=y(0)=1, $\dot{x}(0)=\dot{y}(0)=0$, please solve for the Laplace transform of $x(t)$, i.e. $\mathcal{L}[x(t)]$

4. (15%)Find the angle between the surfaces $xy^2z = 3x+z^2$ and $3x^2-y^2+2z = 1$ at the point (1, -2, 1).

5. (20%)Consider a long thin bar of constant cross section and homogeneous material which is oriented along the x-axis and is perfectly insulated laterally. The heat equation is thus

$$\frac{\partial T}{\partial t} = c^2 \frac{\partial^2 T}{\partial x^2}$$

where T(x,t) is the temperature in the body and c^2 is the thermal diffusivity of material of the bar. The ends x = 0 and x = L of the bar are kept at temperature zero and the initial temperature in the bar is f(x). By applying the method of separation of variables, find the temperature T(x,t) in the bar.

6. (15%)Consider

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + k^2x^2y = 0, \quad 0 \le x \le R, \ y(0) = \text{finite}, \ \ y(R) = 0,$$

where R is a finite value.

(a) Transform the above equation, in $y(x)$, into $y(z)$ with $z = kx$. Also write out the name of the resultant equation, which is of an important form. (4%)

(b) Write the general solution of the resultant equation, with the undertermined coefficients remaining. (You don't need to go through the power series solution procedure. Just write it out.) (3%)

(c) Applying the boundary conditions, you can obtain solutions only for certain special values, which are called eigenvalues. The solutions corresponding to each eigenvalue are called the eigenfunctions. Write the expressions of the eigenvalues and eigenfunctions. Also describe how you determine them. (You may just use the common notation for the eigenvalues without giving the exact values.) (8%)

7. (10%)Consider $\frac{1}{z^4 - z^5}$, with the complex variable $z = x + iy$.

(a) Write the Laurent series expansions respectively for the two different regions $0 < |z| < 1$ and $|z| > 1$. (6%)

(b) Calculate $\oint_C \frac{dz}{z^4 - z^5}$, where C is the circle of $|z| = \frac{1}{2}$ (counterclockwise). (4%)